Hints to Solutions of Certain Problems in H.W. #3

#8 P67

Equation (3) before Thm 3.2 "requires a fixed number of factors"

#9 P67

\[ y_n = \sqrt{n+1} - \sqrt{n} = \frac{(\sqrt{n+1} - \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} \]

\[ = \frac{1}{\sqrt{n+1} + \sqrt{n}} \]

\[ \sqrt{n} y_n = \frac{\sqrt{n}}{\sqrt{n+1} + \sqrt{n}} \]

*10a \( (3\sqrt{n})^{\frac{1}{n}} = (\sqrt{3})^n (\sqrt[2n]{n}) = (\sqrt{3}) \left(\sqrt[n]{n} \right)^{\frac{1}{2}} \) and use established results and theorems.

14) \[ b < (a^n + b^n) < 2b^n \]

\[ b < (a^n + b^n)^{\frac{1}{n}} < (2)^{\frac{1}{2}} b \]

and apply squeeze theorem.

P 74

#3 \( x_1 > 2 \). Suppose \( x_k > 2 \), then

\[ x_{k+1} = 1 + \sqrt{x_k - 1} \geq 1 + \sqrt{2 - 1} = 2 \]

So by induction \( x_k > 2 \) for all \( k \geq 1 \).

\[ x_{k+1} - x_k = 1 + \sqrt{x_k - 1} - x_k = \frac{1}{\sqrt{x_k - 1} - 1} = \frac{1}{x_k - 1} (1 - x_k - 1) \]

Since \( x_k > 2 \), second factor is 0 or negative. \( x_k \) sequence is decreasing.

To find limit, let \( x_\infty = 2 \).
\[ y_{n+1} - y_n = \frac{1}{2n} + \frac{1}{2n-2} - \frac{1}{n+1} = \frac{1}{2n+1} - \frac{1}{2(n+1)} > 0 \]
\[ y_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} < \frac{1}{n+1} + \cdots + \frac{1}{n+1} = \frac{n}{n+1} \leq 1 \]

\[ (1 + \frac{1}{n+1})^n = (1 + \frac{1}{n})^{n+1} (1 + \frac{1}{n})^{-1} \]

\[ \text{P}80 \]

\# 6

\[ y_{n+1} < y_n \iff (n+1)^{\frac{1}{n+1}} < n^{\frac{1}{n}} \iff n^{\frac{n+1}{n+1}} (1 + \frac{1}{n})^{\frac{1}{n}} < n^{\frac{n}{n}} \iff n^{(1 + \frac{1}{n})} < n^{(1 + \frac{1}{n})} \]

\[ \iff (1 + \frac{1}{n}) > n^{\frac{1}{n}} \iff (1 + \frac{1}{n}) > n. \]

Inequality \( (1 + \frac{1}{n})^n < n \). True for \( n = 3 \).

Suppose true for \( k \). Then

\[ (1 + \frac{1}{k+1})^{k+1} < (1 + \frac{1}{k})^{k+1} \leq (1 + \frac{1}{k})^{k+1} \leq \frac{k}{k+1} \]

\[ \text{inductive hypothesis} \]

(b) Subsequence \( (2n)^{\frac{1}{2n}} \) converges to same limit, say \( x \). As does \( n^{\frac{1}{n}} \)

Now:

\[ (2n)^{\frac{1}{2n}} (\frac{1}{2})^{\frac{1}{2n}} \left[ \left( n^{\frac{1}{n}} \right) \right]^{\frac{1}{2}} \rightarrow (1)(x^{\frac{1}{2}}) \]

\[ x = x^{\frac{1}{2}} \quad \text{and} \quad x = 0 \text{ or } x = 1 \]

Hence since \( x_n > 1, x = 1 \)

(c) \( (1 + \frac{1}{n})^{2n^2} = \left[ \left( \frac{1 + \frac{1}{n}}{n^2} \right)^n \right]^2 \)

Since \( n^{\frac{1}{n}} \text{ is a subsequence of } n^{\frac{1}{n}} \)

\[ (1 + \frac{1}{n})^n \rightarrow e \quad \text{and} \quad (1 + \frac{1}{n^2})^{2n^2} \rightarrow e^2 \]