Purdue University MA 173
Calculus and Analytic Geometry II
Fall 2003, Test One
(Instructor: Aaron N. K. Yip)

- This test booklet has FIVE QUESTIONS, totaling 50 points for the whole test. You have 50 minutes to do this test. Plan your time well. Read the questions carefully. You do not need to attempt the questions in sequence.

- This test is closed book and note. No calculator is allowed.

- In order to get full credits, you need to give correct and simplified answers and explain in a comprehensible way how you arrive at them.

- You can use both sides of the papers to write your answers. But please indicate so if you do.
1. Consider the straight line segment \( PQ \) with end points \( P \) and \( Q \). Let \( R \) be the point on the segment \( PQ \) such that

\[
\frac{|PR|}{|RQ|} = \frac{\alpha}{\beta}
\]

where \(|\cdot|\) denotes the length of a vector.

Express the vector \( \vec{R} \) in terms of \( \vec{P}, \vec{Q}, \alpha \) and \( \beta \).

(Hint: What is the relationship between \( \vec{PR} \) and \( \vec{RQ} \)?)

\[
\beta |\vec{PR}| = \alpha |\vec{RQ}|
\]

\[
\beta \vec{PR} = \alpha \vec{RQ}
\]

\[
\beta (\vec{R} - \vec{P}) = \alpha (\vec{Q} - \vec{R})
\]

\[
\beta \vec{R} - \beta \vec{P} = \alpha \vec{Q} - \alpha \vec{R}
\]

\[
(\alpha + \beta) \vec{R} = \alpha \vec{Q} + \beta \vec{P}
\]

\[
\vec{R} = \frac{\alpha \vec{Q} + \beta \vec{P}}{\alpha + \beta}
\]
2. Given the triangle ΔABC in \( R^3 \) with: \( A = (1, 2, -1) \), \( B = (1, 1, 1) \) and \( C = (-2, 1, 4) \). Find the cosine of the angle at \( A \) and also the area of ΔABC.

\[
\overrightarrow{AB} = (0, -1, 2), \quad |\overrightarrow{AB}| = \sqrt{1^2 + 2^2} = \sqrt{5}
\]

\[
\overrightarrow{AC} = (-3, -1, 5), \quad |\overrightarrow{AC}| = \sqrt{9 + 1 + 25} = \sqrt{35}
\]

\[
\cos \angle A = \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{|\overrightarrow{AB}| |\overrightarrow{AC}|} = \frac{1 + 10}{\sqrt{5} \sqrt{35}} = \frac{11}{5 \sqrt{7}}.
\]

\[
\text{Area} = \frac{1}{2} |\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2} \left| \begin{array}{ccc} \hat{i} & \hat{j} & \hat{k} \\ 0 & -1 & 2 \\ -3 & -1 & 5 \end{array} \right|
\]

\[
= \frac{1}{2} \left[ \hat{i}(\hat{j}(-1 \cdot 5) - \hat{k}(0 \cdot 2)) + \hat{j}(\hat{i}(-3 \cdot 5) + k(-3 \cdot 2)) \right]
\]

\[
= \frac{1}{2} \left[ -3\hat{i} - 6\hat{j} - 3\hat{k} \right]
\]

\[
\text{Area} = \frac{1}{2} \sqrt{9 + 36 + 9} = \frac{1}{2} \sqrt{54} = \frac{3}{2} \sqrt{6}.
\]
4. Consider the region under the graph $y = 1 - x^2$ and above the $x$-axis. Find the volume of the solid formed by

(a) rotating the region with respect to the $x$-axis.
(b) rotating the region with respect to the $y$-axis.

\[ V = \int_{-1}^{1} \pi y^2 \, dx \]
\[ = \pi \int_{-1}^{1} (1-x^2)^2 \, dx \]
\[ = 2\pi \int_0^1 (1-2x^2+x^4) \, dx = 2\pi \left[ 1 - \frac{2}{3} + \frac{1}{5} \right] = 2\pi \left[ \frac{15 - 10 + 3}{15} \right] \]
\[ = \frac{16\pi}{15} = \frac{16}{15}\pi \]

\[ V = \int_{0}^{1} \pi x^2 \, dy = \int_{0}^{1} \pi (1-y) \, dy = \pi \left[ 1 - \frac{1}{2} \right] \]
\[ = \frac{\pi}{2} \]
5. Compute the following integrals:

(a) \[ \int_0^1 x \sqrt{1 - x^2} \, dx \]

(b) \[ \int_0^{\frac{3}{2}} \sqrt{9 - 4x^2} \, dx \]

(c) \[ \int_{-4}^4 y \sqrt{16 - y^2} \, dy \]

(d) \[ \int_{-4}^4 \sqrt{16 - y^2} \, dy \]

(Hint: You can use the fact that \( \int_0^1 \sqrt{1 - x^2} \, dx = \frac{\pi}{4} \).)

(a) \[ \int_0^1 x \sqrt{1 - x^2} \, dx = \int_0^1 \sqrt{1-u} \, \frac{du}{2} \]

\[ = \frac{1}{2} \left[ (1-u)^{\frac{3}{2}} \right]_{0}^{1} = \frac{1}{3} \]

(b) \[ \int_0^{\frac{3}{2}} \sqrt{9 - 4x^2} \, dx = \int_0^1 \sqrt{9 - 9u^2} \cdot \frac{3}{2} \, dx \]

\[ u = \frac{2x}{3}, \quad du = \frac{2}{3} \, dx \]

\[ \sqrt{9 - 4x^2} = \sqrt{9 - 4 \left( \frac{3u}{2} \right)^2} = \sqrt{9 - 9u^2} \]

\[ = \frac{9}{2} \int_0^1 \sqrt{1-u^2} \, du = \frac{9\pi}{8} \]

(c) \[ \int_{-4}^4 y \sqrt{16 - y^2} \, dy = 0 \quad (\because y \sqrt{16 - y^2} \text{ is an odd function}) \]

 integrates over an symmetric interval.
(c) \[ \int_{-4}^{4} \sqrt{16-y^2} \, dy = 2 \int_{0}^{4} \sqrt{16-u^2} \, du \]

\[ u = \frac{y}{4}, \quad y = 4u \]

\[ dy = 4 \, du \]

\[ = 2 \int_{0}^{1} \sqrt{16-16u^2} \, 4 \, du \]

\[ = 32 \int_{0}^{1} \sqrt{1-u^2} \, du \]

\[ = 32 \left( \frac{u}{2} \right) \bigg|_{0}^{1} \]

\[ = 8\pi \]