Polar Coordinates

Instead of referring to a point \( P \) by its \((x, y)\) coordinates, we can use the origin as a "pole" and the positive \( x\)-axis as a polar axis and refer to a point by a distance and an angle.

That is refer to a point by a distance and direction.
Example:

Find the points

\( (5, \frac{7\pi}{8}) \), \( (2/3, \frac{5\pi}{3}) \),

\( (-1, \frac{\pi}{4}) \), \( (4, \frac{\pi}{6}) \), \( (2, 8\pi/3) \)
Converting between coordinate systems:

If \( P \) has coordinates \((x, y)\)
then its polar coordinates are

\[
\cos \theta = \frac{x}{r} \quad \sin \theta = \frac{y}{r}
\]

\[
x = r \cos \theta
\]

\[
y = r \sin \theta
\]
If we know \( P(x,y) \), then
\[
    r^2 = x^2 + y^2, \quad \tan \theta = \frac{y}{x}
\]

Convert \((2, \sqrt{3})\) to Cartesian coordinates
\[
    x = \quad y =
\]

Convert \((2, 2)\) to polar coordinates
\[
    r = \quad \theta =
\]
A graph of an equation $r = f(\theta)$ is the points $(r, \theta)$ satisfying the given equation:

Ex: $r = 3$

Circle of radius 3 centered at $(0,0)$

Sometimes the equation may be:

$\theta = F(r)$, or $G(r, \theta) = 0$

Ex: $\theta = 7\pi/6$
Example: \( r = 4 \sin \theta \)

Note: Since \( r(\theta + \pi) = 4 \sin(\theta + \pi) = -4 \sin \theta = -r(\theta) \),
the point \( (-r, \theta + \pi) = (r, \theta) \).
So we can find the graph by looking at \( 0 \leq \theta \leq \pi \).
Is it really a circle?

\[ r = 4 \sin \theta; \quad y = r \sin \theta \]
\[ \frac{r}{4} = \sin \theta = \frac{y}{r} \quad \Rightarrow \]
\[ r^2 = 4y \; ; \quad \text{But } r^2 = x^2 + y^2 \; ; \]

So \[ x^2 + y^2 = 4y \quad \Rightarrow \]
\[ x^2 + y^2 - 4y = 0 \]
\[ x^2 + (y - 2)^2 = 4 \]

Circle of radius 2, center \((0, 2)\).

\[ r = 2 - \sin \theta \]
\[ r = \sin 2\theta \]
\[ = \frac{3\sqrt{3} - 4}{11} \]

Find the points where the Cardioid

\[ r = 2 - \sin \theta \]

has a horizontal tangent.

\[ \frac{dy}{d\theta} = 2 \cos \theta (1 - \sin \theta) \]

\[ \frac{dy}{d\theta} = 0; \quad \cos \theta = 0, \quad \text{or} \quad \sin \theta = 1 \]

\[ \theta = \frac{\pi}{2}, \quad 3\frac{\pi}{2} \]
Conic Sections

A conic section is obtained by "slicing" a cone by a plane.
Also hyperbolae (Monday)

Here we describe the conic sections by their properties and equations.
A parabola is the set of points in the plane which are equidistant from a fixed point $F$ (focus) and a fixed line $l$ (directrix). The midpoint of the segment from the focus to the directrix is the vertex. The line connecting the focus and vertex is the axis.
4. In order to write the equation of a parabola we first assume the vertex is $(0,0)$, and focus is $(0,p)$. So the directrix is $y = -p$. 

![Parabola Diagram]
The parabola satisfies

\[ |FP| = |PL| \]

If \( P = (x, y) \), then \( |FP| = \sqrt{(x-h)^2 + (y-k)^2} = \sqrt{x^2 + (y-p)^2} \)

On the other hand \( |PL| = 1y+p \)

So the equation of the parabola

is:

\[ \sqrt{x^2 + (y-p)^2} = 1y+p \]

\[ x^2 + (y-p)^2 = (y+p)^2 \]

\[ x^2 + \sqrt{y^2-2py+p^2} + p^2 = y^2 + 2py + p^2 \]

\[ \Rightarrow x^2 = 4py \]

This is the equation of a parabola with focus (0, p) and directrix \( y = -p \).
We can rewrite this as:
\[ y = a x^2 \text{, with } a = \frac{1}{4p} \]

The parabola opens upward (concave up) if \( p > 0 \) (i.e. if \( a > 0 \)) and down if \( p < 0 \). Note the graph must be symmetric with respect to the \( y \)-axis.

If we interchange the roles of \( y \) and \( x \) we get
\[ y^2 = 4px \]

and this is the equation of a parabola with focus \((p, 0)\) and directrix \( x = -p \).
Example: Find the directrix, focus, and vertex of \( y^2 = 12x \)

This is in the form \( y^2 = 4px \), with \( p = 3 \). So the focus is \((3,0)\), directrix, \( x = -3 \), and vertex \((0,0)\)
We can shift the focus and directrix in the "obvious" way.

If the vertex is \((h, k)\) and the directrix is \(y = k - p\), then the focus is \((h, k + p)\) and the equation is:

\[
(X - h)^2 = 4p(Y - k)
\]
An ellipse is the set of points in the plane whose distances from two fixed points \( F_1, F_2 \) (foci) is a constant.

Let's again start with a simple case. Suppose the foci are at \((-c, 0)\) and \((c, 0)\).
Setting $F_1 = (-c,0)$, $F_2 = (c,0)$ and letting the constant distance which is the sum $(PF_1 + PF_2)$ be $2a$, we get

$$|PF_1| + |PF_2| = 2a \Rightarrow 
\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a$$

This equation takes more work to simplify than that for a parabola (pp. 677).

Eventually we get: $(a^2-c^2)x^2 + a^2y^2 = a^2(a^2-c^2)$

Note

$$C < a \quad \text{let} \quad b^2 = a^2 - c^2
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$
The equation

\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 , \]

with \( a > b > 0 \) is the equation of an ellipse with foci \(( \pm c, 0)\), with \( c^2 = a^2 - b^2 \)

Switching roles of \((x,y)\) we see

\[ \frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 , \quad a > b > 0 \]

is an ellipse with foci \((0, \pm c)\), where \( c^2 = a^2 - b^2 \).
For equation (2) the points 
\((-a, 0)\) and \((a, 0)\) are called 
the **vertices** of the ellipse. 

The line joining the vertices is the 
**major axis**.

(Similar for Eq. (3) with x/y roles switched)
**Ex:** Find vertices, foci, and sketch: \( 25x^2 + 9y^2 = 225 \)

\[
\frac{x^2}{9} + \frac{y^2}{25} = 1;
\]

\( a = 5, \quad b = 3, \quad c^2 = 25 - 9 = 16, \quad c = 4. \)

**Foci:** \((0, \pm 4)\); **Vertices:** \((0, \pm 5)\)
Suppose we know the foci are $(\pm 5, 0)$, and vertices are $(\pm 13, 0)$.

What is the equation, and sketch?

e = 5, \quad a = 13, \quad \text{so}

\[ b^2 = a^2 - c^2 = 13^2 - 5^2 = 12^2 \]

So,

\[ \frac{x^2}{(13)^2} + \frac{y^2}{(12)^2} = 1 \]

\[ \frac{x^2}{169} + \frac{y^2}{144} = 1 \]
If the foci are: 

$(h+c, K)$ and 

vertices $(h+a, K)$, then 

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

$b^2 = c^2 - a^2$. 

![Diagram of an ellipse with foci and vertices labeled]