POWER SERIES

A **POWER SERIES** is

A SERIES OF THE FORM

\[
\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n + \ldots
\]

WITH \(x\) A VARIABLE.

THE CONSTANTS \(\{c_n\}\) ARE CALLED THE COEFFICIENTS OF THE SERIES

FOR EACH \(x\) WE CAN ASK IF (1) CONVERGES OR DIVERGES
WE ALREADY KNOW MANY POWER SERIES:

\[ \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \ldots \]

\[ = \frac{1}{1 - x} \quad -1 < x < 1 \]

and diverges otherwise

(3) ANY POLYNOMIAL IS A POWER SERIES

\[ c_0 + c_1 x + \ldots + c_n x^n + o(x^n) + \ldots \]

i.e. all but finite number of \( c_i \) are zero
A power series centered at $x = a$:

$$\sum_{n=0}^{\infty} C_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \ldots + c_n (x-a)^n + \ldots$$

Also called a power series about $a$.

Example: Find all $x$ for which $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges.
USE RATIO TEST:
\[ a_n = \frac{x^n}{n!} \]

Then
\[ \left| \frac{a_{n+1}}{a_n} \right| = \frac{|x|^{n+1}}{(n+1)!} \cdot \frac{n!}{|x|^n} = \frac{|x|}{n+1} ; \quad \text{So} \]

\[ \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 0 < 1, \quad \text{Thus} \]

\[ \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{CONVERGES} \]

FOR ALL \( x \).

HOW ABOUT
\[ \sum_{n=0}^{\infty} \frac{n^2 x^n}{10^n} ? \]
Again we use the **Ratio Test**: 

\[
\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)^2 |x|^{n+1}}{10^n} \cdot \frac{10^n}{n^2 |x|^n}
\]

\[= \left(\frac{n+1}{n}\right)^2 \cdot \frac{|x|}{10}\]

\[
\lim_{n \to \infty} \left(\frac{|a_{n+1}|}{|a_n|}\right) = \lim_{n \to \infty} \left(\frac{n+1}{n}\right)^2 \frac{|x|}{10}
\]

\[= \frac{|x|}{10}.
\]

So the series converges absolutely if \(\frac{|x|}{10} < 1\)

i.e. \(-10 < x < 10\),

and diverges for \(|x| > 10\).

For \(x = \pm 10\) we must check explicitly.
That is because the ratio test is inconclusive when

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1.$$ 

$$X = 10; \quad \sum_{n=1}^{\infty} \frac{n^2 \cdot 10^n}{10^n} = \sum_{n=1}^{\infty} n^2$$

diverges by the divergence test: $$\lim_{n \to \infty} n^2 = \infty.$$ 

$$X = -10, \quad \sum_{n=1}^{\infty} \frac{n^2 \cdot (-10)^n}{10^n} = \sum_{n=1}^{\infty} (-1)^n n^2$$
also diverges, since $$\lim_{n \to \infty} (-1)^n n^2$$ doesn't exist.

Ex: $$\sum_{n=1}^{\infty} \frac{(x-4)^n}{2n}.$$ For what $$x$$ does the series converge?
\[ |\frac{a_{n+1}}{a_n}| = \frac{|x-4|^{n+1}}{2(n+1)} \cdot \frac{2n}{|x-4|^n} \leq R \text{ Test} \]

\[ = |x-4| \cdot \frac{n}{n+1} \]

So \( \lim_{n \to \infty} |\frac{a_{n+1}}{a_n}| = |x-4| \)

For \( |x-4| < 1 \), \( \sum_{n=0}^{\infty} \frac{(x-4)^n}{2^n} \)

CONVERGES ABSOLUTELY

\(-1 < x-4 < 1\)
\[ 3 < x < 5 \]

For \( x > 5 \) or \( x < 3 \)

\((|x-4| > 1) \) THE SERIES

DIVERGES.

\[ x = 3; \quad \sum_{n=0}^{\infty} \frac{(3-4)^n}{2^n} \]

\[ = \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \]

CONVERGES (CONDITIONALLY)

\[ x = 5; \quad \sum_{n=1}^{\infty} \frac{(5-4)^n}{2^n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{2^n} \]

DIVERGES.
THEOREM:

Given a power series \[ \sum_{n=0}^{\infty} C_n(x-a)^n \] there are 3 possibilities:

(i) The series converges only when \( x = a \)

(ii) The series converges for all \( x \)

(iii) There is a positive number \( R \) so that the series converges if \( |x-a| < R \) and diverges if \( |x-a| > R \)

We call \( R \) the radius of convergence.

\[ a - R < x < a + R \]
In (ii) we say the radius of convergence is $a$; and in (i) the radius of convergence is 0.

Interval of convergence:

(iii) \begin{align*}
(a-R, a+R) \\
(a-R, a+R] \\
[a-R, a+R) \\
[a-R, a+R]
\end{align*}

possibilities depends on convergence at endpoints

(i) Interval $[a]$ \\
(ii) Interval $(-\infty, \infty)$
**Examples:**

**Geometric series:**
\[
\sum_{n=0}^{\infty} x^n \quad R = 1, \quad (-1, 1)
\]

Any polynomial
\[
c_0 + c_1 x + \cdots + c_n x^n \quad R = \infty, \quad (-\infty, \infty)
\]

\[
\sum_{n=0}^{\infty} \frac{x^n}{n!} \quad R = \infty, \quad (-\infty, \infty)
\]

\[
\sum_{n=0}^{\infty} \frac{n^2 x^n}{10^n} \quad R = 10 \quad (-10, 10)
\]

\[
\sum \frac{(x-4)^n}{2^n} \quad R = 1 \quad (-3, 5)
\]