MA 511, Session 1

**Linear Systems**

A basic problem in linear algebra is that of solving the linear system

\[
\begin{align*}
\begin{cases}
    a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
    a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
    \cdots & \cdots \\
    a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{cases}
\end{align*}
\]

This is a system of \( m \) **linear equations** in \( n \) unknowns. The \( a_{ij} \)'s and the \( b_j \)'s are constants called, respectively, the coefficients of the system and the right-hand sides (or forcing terms). If \( b_j = 0 \) for \( 1 \leq j \leq m \), the system is **homogeneous**.

The **locus** of points satisfying the equation

\[
a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i
\]

is an \((n - 1)\)-dimensional plane in the \( n \)-dimensional space \( \mathbb{R}^n \).
Example: For \( m = n = 2 \), a system

\[
\begin{align*}
  x + y &= 1 \\
  -x + y &= 0
\end{align*}
\]

has a solution \((x, y) = (\frac{1}{2}, \frac{1}{2})\), which is the intersection point of the lines \( x + y = 1 \) and \( x = y \).

Example: For \( m = n = 3 \), a system

\[
\begin{align*}
  2x - y &= 0 \\
  -x + 2y - z &= 1 \\
  -y + 2z &= 2
\end{align*}
\]

has a solution \((x, y, z) = (1, 2, 2)\) which is the intersection point of the planes \( 2x - y = 0 \), \(-x + 2y - z = 1\) and \(-2y + z = 2\).
In parametric form a line can be described by one parameter (in a space of any number of dimensions $n$)

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{pmatrix} = t \begin{pmatrix} c_1 \\ c_2 \\ \cdot \\ \cdot \\ c_n \end{pmatrix} + \begin{pmatrix} d_1 \\ d_2 \\ \cdot \\ \cdot \\ d_n \end{pmatrix},$$

where $\overrightarrow{c} = (c_1, c_2, \ldots, c_n)^T$ is a vector in the direction of the line and $\overrightarrow{d} = (d_1, d_2, \ldots, d_n)^T$ is a point on the line.
Similarly, a plane can be described by two parameters (in a space of any number of dimensions \(n\))

\[
X = \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{pmatrix} = s \begin{pmatrix} c_1 \\ c_2 \\ \cdot \\ \cdot \\ c_n \end{pmatrix} + t \begin{pmatrix} d_1 \\ d_2 \\ \cdot \\ \cdot \\ d_n \end{pmatrix} + \begin{pmatrix} e_1 \\ e_2 \\ \cdot \\ \cdot \\ e_n \end{pmatrix}
\]

Here \(\vec{c} = (c_1, c_2, \ldots, c_n)^T\) and \(\vec{d} = (d_1, d_2, \ldots, d_n)^T\) are vectors in the plane with different directions, and \(\vec{e} = (e_1, e_2, \ldots, e_n)^T\) is a point on the plane.
Let us introduce the following notation for the columns of the array \((a_{ij})\) and for the right-hand sides in (1):

\[
\vec{c}_i = \begin{pmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{pmatrix}, \quad 1 \leq i \leq n, \quad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.
\]

Then, the system (1) can be re-written as

\[
x_1 \vec{c}_1 + x_2 \vec{c}_2 + \cdots + x_n \vec{c}_n = \vec{b}.
\]

We immediately see that the system is consistent (i.e., it admits solution) if and only if \(\vec{b}\) is a combination of the columns of the array \((a_{ij})\) (algebraic approach).

Looking at the rows of (1), we see the system is consistent if, and only if the \(m\) planes intersect (geometric approach).
Gaussian Elimination

Example: Consider in $\mathbb{R}^3$ the three planes given by

\[
\begin{align*}
  x + 2y + 3z &= 1 \\
  4x + 5y + 6z &= 4 \\
  7x + 8y + 9z &= 8
\end{align*}
\]

Then,

\[
\begin{align*}
  x + 2y + 3z &= 1 \\
  0x - 3y - 6z &= 0 \\
  0x - 6y - 12z &= 1
\end{align*}
\]

and

\[
\begin{align*}
  x + 2y + 3z &= 1 \\
  0x + y + 2z &= 0 \\
  0x + y + 2z &= -\frac{1}{6}
\end{align*}
\]

We see the system is inconsistent since the second and third equations cannot be satisfied simultaneously.
If we change the 8 on the right of the third equation to 7, then we have

\[
\begin{cases}
x + 2y + 3z & = 1 \\
4x + 5y + 6z & = 4 \\
7x + 8y + 9z & = 7
\end{cases}
\]

or

\[
\begin{cases}
x + 2y + 3z & = 1 \\
0x - 3y - 6z & = 0 \\
0x - 6y - 12z & = 0
\end{cases}
\]

and

\[
\begin{cases}
x + 2y + 3z & = 1 \\
0x + y + 2z & = 0 \\
0x + y + 2z & = 0
\end{cases}
\]

so that

\[
\begin{cases}
x + 2y + 3z & = 1 \\
0x + y + 2z & = 0
\end{cases}
\]

We now see the locus of solutions consists of the line

\[z = t, \quad y = -2t, \quad x = 1 + t.\]
Similarly, if the 9 in the original system were changed to any other number \( r \), then the intersection of the three planes would be a single point:

\[
\begin{align*}
    z &= \frac{1}{r-9}, \\
    y &= -\frac{2}{r-9}, \\
    x &= \frac{r-8}{r-9}.
\end{align*}
\]

Gaussian elimination provides a systematic way of doing such analysis.

This involves two distinct parts: *elimination* by row operations, and *back substitution*.

Three types of elementary row operations are permitted that produce *equivalent* systems (i.e. systems with exactly the same solutions):

- Interchanging two rows
- Multiplying a row by a non-zero number
- Replacing a row by itself plus a multiple of another row