LECTURE 6

6. OPTIMAL REGULARITY (CONTINUED)

6.1. OBSTACLE-TYPE PROBLEMS. In this section, following the idea of Shahgholian, we use the Caffarelli-Jerison-Kenig estimate from the previous lecture to prove the optimal regularity in “no-sign”, superconductivity and two-phase obstacle problem. In fact, we place these equations into a more general framework and establish the $C^{1,1}$ regularity there.

Namely, suppose that we are given a function $u \in W^{2,p}(D)$, $p > n$, which satisfies
\begin{equation}
\Delta u = g \quad \text{in} \quad D,
\end{equation}
in a sense of distribution for some $g \in L^\infty(D)$. Suppose further that there exist an open subset $G \subset D$ such that
\begin{equation}
|\nabla u| = 0 \quad \text{in} \quad D \setminus G
\end{equation}
and in $G$ the right-hand side is given by
\begin{equation}
g(x) = f(x, u(x)), \quad x \in G,
\end{equation}
where $f : G \times \mathbb{R} \to \mathbb{R}$ satisfies the following structural conditions: there exists $M_1, M_2 > 0$ such that
\begin{equation}
|f(x, t) - f(y, t)| \leq M_1|x - y|, \quad x, y \in G, \quad t \in \mathbb{R}
\end{equation}
\begin{equation}
f(x, s) - f(x, t) \geq -M_2(s - t), \quad x \in G, \quad s, t \in \mathbb{R}, \quad s \geq t.
\end{equation}
Locally, these conditions are equivalent to
\begin{equation}
|\nabla_x f(x, t)| \leq M_1, \quad \partial_t f(x, t) \geq -M_2
\end{equation}
in the sense of distributions.

Let us now see how Problems A–C fit into this framework.

- **Problem A**: No-sign obstacle problem
  \[ \Delta u = f(x)\chi_\Omega, \quad \Omega = D \setminus \{u = |\nabla u| = 0\} \]
  with $f \in C^{0,1}(D)$. Here we take $G = \Omega$.

- **Problem B**: Superconductivity problem
  \[ \Delta u = f(x)\chi_\Omega, \quad \Omega = \{|\nabla u| > 0\}. \]
  with $f(x) \in C^{0,1}(D)$. In this problem we also take $G = \Omega$.

- **Problem C**: Two-phase obstacle problem
  \[ \Delta u = \lambda_+\chi_{\{u > 0\}} - \lambda_-\chi_{\{u < 0\}} \quad \text{with} \quad \lambda_+ + \lambda_- \geq 0. \]
  Here we take $G = D$.

**Theorem 6.1** ($C^{1,1}$-regularity). Let $u \in L^\infty(D)$ satisfy (6.1)–(6.5). Then $u \in C^{1,1}_{\text{loc}}(D)$ and
\[ \|u\|_{C^{1,1}(K)} \leq CM \left(1 + \|u\|_{L^\infty(D)} + \|g\|_{L^\infty(D)}\right), \]
for any open $K \subset D$, where $C = C(n, K, D)$ and $M = \max\{1, M_1, M_2\}$. 

The proof is based on the following lemma which is a direct consequence of the structural assumptions on $f$ and $u$.

**Lemma 6.2.** Let $u \in C^1(D)$ satisfy (6.1)-(6.5). Then for any unit vector $e$,
\[
\Delta(\partial_e u) \geq -L \quad \text{in } D,
\]
where $L = M_1 + M_2 \|\nabla u\|_{L^\infty(D)}$.

**Proof.** Fix a direction $e$ and let $v = \partial_e u$. Let also
\[
E := \{v > 0\}.
\]
Note that $E \subset G$ because of the assumption (6.2). Then, formally, for $x \in E$,
\[
\Delta(v^+) = \partial_e \Delta u(x) = e \cdot \nabla_x f(x, u) + \partial_t f(x, u) Deu \geq -M_1 - M_2 \|\nabla u\|_{L^\infty(D)} =: -L.
\]
To justify this computation, observe that $\Delta(v^+) \geq -L$ in $D$ is equivalent to the inequality
\[
(6.6) \quad -\int_D \nabla(v^+) \nabla \eta \, dx \geq -L \int_D \eta \, dx
\]
for any nonnegative $\eta \in C^\infty_0(D)$. Suppose first that $\text{supp} \eta \subset \{v > \delta\}$ with $\delta > 0$. Then writing the equation
\[
-\int_D \nabla u \nabla \eta \, dx = \int_D f \eta \, dx
\]
with $\eta = \eta(x)$ and $\eta = \eta(x - he)$, we obtain an equation for the incremental quotient
\[
v_h(x) := \frac{u(x + he) - u(x)}{h}.
\]
Namely, we obtain
\[
(6.7) \quad -\int_D \nabla v_h \nabla \eta \, dx = \frac{1}{h} \int_D [f(x + he, u(x + he)) - f(x, u(x))] \eta \, dx
\]
for small $h > 0$. Note that $u(x + he) > u(x)$ on $\text{supp} \eta \subset \{v > \delta\}$ and from the hypotheses on $f$ we have
\[
f(x + he, u(x + he)) - f(x, u(x))
\geq [f(x + he, u(x + he)) - f(x + he, u(x))] + [f(x + he, u(x)) - f(x, u(x))]
\geq -M_1 h - M_2 |u(x + he) - u(x)|
\]
for small $h$. Letting in (6.7) $h \to 0$ and then $\delta \to 0$ we arrive at
\[
-\int_D \nabla v \nabla \eta \, dx \geq -\int_D (M_1 + M_2 \eta) \, dx
\geq -L \int_D \eta \, dx
\]
for arbitrary $\eta \geq 0$ with $\text{supp} \eta \subset \{v > 0\}$.

Thus, we proved that $\Delta v \geq -L$ in the open set $E = \{v > 0\}$ in the sense of distributions. Then it is a simple exercise to show that (6.6) holds for any nonnegative $\eta \in C^\infty_0(D)$.

To prove the same inequality for $v^-$, we simply reverse the direction $e$.  \qed
Proof of Theorem 6.1. We start by observation that $u$ is twice differentiable a.e. in $D$, since $u \in W^{2,p}_{loc}(D)$ with $p > n$, see e.g. Theorem 1.72 in [Maly-Ziemer]. Then fix a point $x_0 \in K \subset D$ where $u$ is twice differentiable and define

$$v(x) = \partial_e u(x)$$

for a unit vector $e$ orthogonal to $\nabla u(x_0)$ (if $\nabla u(x_0) = 0$, take arbitrary unit $e$).

Without loss of generality we may assume $x_0 = 0$. Our aim is to obtain a uniform estimate for $\partial x_j e^u(0) = \partial x_j v(0)$, $j = 1, \ldots, n$. By construction, $v(0) = 0$ and $v$ is differentiable at 0. Hence, we have the Taylor expansion

$$v(x) = \xi \cdot x + o(|x|), \quad \xi = \nabla v(0).$$

Now, if $\xi = 0$ then $\partial x_j v(0) = 0$ for all $j = 1, \ldots, n$ and there is nothing to estimate. If $\xi \neq 0$, consider the cone

$$\Gamma = \{ x \in \mathbb{R}^n : \xi \cdot x \geq |\xi| |x|/2 \},$$

which has a property that

$$\Gamma \cap B_r \subset \{ v > 0 \}, \quad -\Gamma \cap B_r \subset \{ v < 0 \}$$

for sufficiently small $r > 0$. Consider also the rescalings

$$v_r(x) = \frac{v(rx)}{r}, \quad x \in B_1.$$  

Note that $v_r(x) \to v_0(x) := \xi \cdot x$ uniformly in $B_1$ and consequently $\nabla v^\pm_r \rightharpoonup \nabla v^\pm_0$ weakly in $L^2(B_1)$. Then by Fatou’s lemma, we have

$$c|\xi|^4 = \int_{\Gamma \cap B_1} \frac{\nabla v^+_0(x)^2 dx}{|x|^{n-2}} \int_{\Gamma \cap B_1} \frac{\nabla v^-_0(x)^2 dx}{|x|^{n-2}} \leq \liminf_{r \to 0} \frac{1}{r^4} \int_{\Gamma \cap B_r} \frac{\nabla v^+_r(x)^2 dx}{|x|^{n-2}} \int_{-\Gamma \cap B_r} \frac{\nabla v^-_r(x)^2 dx}{|x|^{n-2}} \leq \liminf_{r \to 0} \Phi(r, v^+, v^-),$$

where $\Phi$ is as in ACF Monotonicity Formula (see Lecture 5). In the next step we apply the CJK estimate (see Lecture 5), however we should suitably adjust (scale) $v^\pm$ first. Let now $\delta = \frac{1}{4} \text{dist}(K, \partial D)$ and $K_\delta = \{ \text{dist}(\cdot, K) < \delta \}$. By Lemma 6.2, we have $\Delta v^\pm \geq -L_\delta$ in $B_\delta(x_0) \subset K_\delta$, where $L_\delta = M_1 + M_2\|\nabla u\|_{L^\infty(K_\delta)}$. Then it is easy to check that the rescalings

$$w^\pm(x) = \frac{v^\pm(\delta x)}{L_\delta \delta^2}, \quad x \in B_1$$
satisfy all hypotheses in CJK estimate. Hence, we have
\[
c|\xi|^4 \leq \liminf_{r \to 0} \Phi(r, v^+, v^-) = CL_\delta^4 \lim_{r \to 0} \Phi(r, w^+, w^-)
\]
\[
\leq CL_\delta^4 \left( 1 + \|w^+\|_{L^2(B_1)}^2 + \|w^-\|_{L^2(B_1)}^2 \right)^2
\]
\[
\leq CL_\delta^4 \left( 1 + \frac{\|\nabla u\|_{L^\infty(K_\delta)}^2}{L_\delta^4} \right)^2
\]
\[
\leq C \left( L_\delta^2 \|\nabla u\|_{L^\infty(K_\delta)}^2 + \frac{\|\nabla u\|_{L^\infty(K_\delta)}^2}{\delta^2} \right)^2 \leq CL^4,
\]
where \( C = C(n, K, D) \) and
\[
L = M(1 + \|\nabla u\|_{L^\infty(K_\delta)}) \leq C\left(\|u\|_{L^\infty(D)} + \|g\|_{L^\infty(D)}\right)
\]
with \( M = \max\{1, M_1, M_2\} \). Recalling now that \( \xi = \nabla \partial_e u(x_0) \), we arrive at
\[
|\nabla \partial_e u(x_0)| \leq CL.
\]
This doesn’t give the desired estimate on \(|D^2 u|\) yet, since \( e \) is subject to the condition \( e \cdot \nabla u(x_0) = 0 \), unless \( \nabla u(x_0) = 0 \). If \( \nabla u(x_0) \neq 0 \), choose the coordinate system so that \( \nabla u(x_0) \) is parallel to \( e_1 \). Then, taking \( e = e_2, \ldots, e_n \) in the estimate above, we obtain
\[
|\partial_{x_i x_j} u(x_0)| \leq CL, \quad i = 2, \ldots, n, \quad j = 1, 2, \ldots, n
\]
To obtain the estimate in the missing direction \( e_1 \), we use the equation \( \Delta u = g \):
\[
|\partial_{x_1 x_1} u(x_0)| \leq |\Delta u(x_0)| + |\partial_{x_2 x_2} u(x_0)| + \ldots + |\partial_{x_n x_n} u(x_0)|
\]
\[
\leq \|g\|_{L^\infty(D)} + CL.
\]
This completes the proof of the theorem. \( \square \)