8. Preliminary Analysis of the Free Boundary

In this lecture we start the analysis of the free boundary. Our main focus is on the obstacle-type problems that we stated in Lecture 2. Namely, we study the following three problems

**Problem A:** \( \Delta u = \chi_\Omega \) in \( D \), \( \Omega = D \setminus \{ u = |\nabla u| = 0 \} \)

**Problem B:** \( \Delta u = \chi_\Omega \) in \( D \), \( \Omega = D \setminus \{ |\nabla u| = 0 \} \)

**Problem C:** \( \Delta u = \lambda_+ \chi_{\{u>0\}} - \lambda_- \chi_{\{u<0\}} \) in \( D \), \( \lambda_\pm > 0 \).

Here \( D \) is a domain in \( \mathbb{R}^n \) and we will assume that the solution \( u \in C^{1,1}(D) \) throughout this lecture and that the equations are satisfied in the a.e. sense.

8.1. Nondegeneracy. The first property we discuss is the nondegeneracy property of the solutions, which is in a sense opposite to \( C^{1,1} \) estimates. This is going to be important when studying the blowups of the solutions.

In Problems A, B, for any \( x_0 \in \Gamma = \Omega(u) \cap D \) we have the estimate

\[
\sup_{B_r(x_0)} u \leq u(x_0) + \frac{M^2}{2} |x - x_0|^2,
\]

if \( B_r(x_0) \subset \subset D \), where \( M = \|D^2u\|_{L^\infty(D)} \). However, the \( C^{1,1} \) regularity doesn’t exclude that \( u(x) - u(x_0) \) will decay faster than quadratically at \( x_0 \). This is taken care of by the following lemma.

**Lemma 8.1** (Nondegeneracy: Problem A). Let \( u \) be a solution of Problem A in \( D \). Then we have the inequality

\[
(8.1) \quad \sup_{\partial B_r(x_0)} u \geq u(x_0) + \frac{r^2}{8n}, \quad \text{for any } x_0 \in \overline{\Omega(u)}
\]

provided \( B_r(x_0) \subset \subset D \).

**Remark 8.2.** Since \( u \) is subharmonic if it solves Problem A, we can replace sup over \( \partial B_r(x_0) \) to one over \( B_r(x_0) \), obtaining an equivalent statement.

Before giving the proof, consider a similar nodegeneracy statement for solutions of \( \Delta u = 1 \).

**Lemma 8.3.** Let \( u \) satisfy \( \Delta u = 1 \) in a ball \( B_R \). Then

\[
\sup_{\partial B_r} u \geq u(0) + \frac{r^2}{2n}, \quad 0 < r < R
\]

**Proof.** Consider the auxiliary function

\[
w(x) = u(x) - \frac{|x|^2}{2n}, \quad x \in B_R.
\]
Then $w$ is harmonic in $B_R$. Therefore by the maximum principle we obtain that

$$w(0) \leq \sup_{\partial B_r} w = \left( \sup_{\partial B_r} u \right) - \frac{r^2}{2n},$$

which implies the required inequality. 

**Proof of Lemma 8.1.** 1) Assume first that $x_0 \in \Omega(u)$ and moreover $u(x_0) > 0$. Consider then the auxiliary function

$$(8.2) \quad w(x) = u(x) - u(x_0) - \frac{|x-x_0|^2}{2n},$$

similar to the one in the proof the previous lemma. We have $\Delta w = 0$ in $B_r(x_0) \cap \Omega$. Since $w(x_0) = 0$, by the maximum principle we have that

$$\sup_{\partial B_r(x_0) \cap \Omega} w \geq 0.$$

Besides, $w(x) = -u(x_0) - |x-x_0|^2/(2n) < 0$ on $\partial \Omega$. Therefore, we must have

$$\sup_{\partial B_r(x_0) \cap \Omega} w \geq 0.$$

The latter is equivalent to

$$\sup_{\partial B_r(x_0) \cap \Omega} u \geq u(x_0) + \frac{r^2}{2n}$$

and the lemma is proved in this case.

2) Suppose now $x_0 \in \Omega(u)$ and $u(x_0) \leq 0$. If $B_{r/2}(x_0)$ contains a point $x_1$ such that $u(x_1) > 0$, then

$$\frac{r^2}{2n} \geq u(x_0) + \frac{r^2}{2n}.$$

If it happens that $u \leq 0$ in $B_{r/2}(x_0)$, from subharmonicity of $u$ and the strong maximum principle we will have that either $u = 0$ identically in $B_{r/2}(x_0)$, or $u < 0$ in $B_{r/2}(x_0)$. The former case is impossible, as $x_0 \in \Omega(u)$, and the latter case implies that $B_{r/2}(x_0) \subset \Omega(u)$ and therefore $\Delta u = 1$ in $B_{r/2}(x_0)$. Then Lemma 8.3 finishes the proof in this case and we obtain

$$\sup_{B_r(x_0)} u \geq \sup_{B_{r/2}(x_0)} u \geq u(x_0) + \frac{r^2}{8n}.$$

3) Finally, for $x_0 \in \overline{\Omega(u)}$, we take a sequence $x_n \in \Omega(u)$ such that $x_n \to x_0$ and pass to the limit in the corresponding nondegeneracy inequality at $x_n$. 

Even though the proof above does not work for Problem B in general, we still have the nondegeneracy.

**Lemma 8.4 (Nondegeneracy: Problem B).** Let $u$ be a solution of Problem B in $D$. Then we have the inequality

$$(8.3) \quad \sup_{\partial B_r(x_0)} u \geq u(x_0) + \frac{r^2}{2n}, \quad \text{for any } x_0 \in \overline{\Omega(u)}$$

provided $B_r(x_0) \subset \subset D$. 

Proof. By continuity, it suffices to obtain the estimate (8.3) only for points \( x_0 \in \Omega(u) \). Note that at those points we have \( |\nabla u(x_0)| \neq 0 \). Consider now the same auxiliary function \( w \) as in (8.2). Then we claim

\[
(8.4) \quad \sup_{B_r(x_0)} w = \sup_{\partial B_r(x_0)} w.
\]

Indeed, if this fails, then the supremum of \( w \) is attained at some interior point \( y \in B_r(x_0) \), and we would have that \( |\nabla w(y)| = 0 \), which implies that

\[
|\nabla u(y)| = \frac{|y - x_0|}{n}.
\]

First off, this implies that \( y \neq x_0 \), otherwise we would have \( |\nabla u(x_0)| = 0 \). Therefore \( |\nabla u(y)| > 0 \) and consequently \( y \in \Omega(u) \). Since \( w \) is harmonic in \( \Omega(u) \), by the strong maximum principle \( w \) is constant in a neighborhood of \( y \). Thus, the set of maxima of \( w \) is both relatively open and closed in \( B_r(x_0) \), which implies that \( w \) is constant there and (8.4) is trivially satisfied. \( \square \)

Finally, in Problem C we have nondegeneracy in both phases, provided \( \lambda_\pm > 0 \).

**Lemma 8.5** (Nondegeneracy: Problem C). If \( u \) is a solution of Problem C in \( D \), then we have

\[
(8.5) \quad \sup_{\partial B_r(x_0)} u \geq u(x_0) + \lambda_+ \frac{r^2}{2n}, \quad \text{for any } x_0 \in \Omega_+(u)
\]

\[
(8.6) \quad \inf_{\partial B_r(x_0)} u \leq u(x_0) - \lambda_- \frac{r^2}{2n}, \quad \text{for any } x_0 \in \Omega_-(u)
\]

provided \( B_r(x_0) \subset D \).

**Proof.** To prove these inequalities, we consider the auxiliary functions

\[
w(x) = u(x) - u(x_0) - \lambda_\pm \frac{|x - x_0|^2}{2n}
\]

and argue similarly to part 1) of the proof of Lemma 8.1. We leave the details to the reader. \( \square \)

**Corollary 8.6** (Nondegeneracy of the gradient). Under the conditions of Lemmas 8.1, 8.4, 8.5 the following inequality holds

\[
\sup_{B_r(x_0)} |\nabla u| \geq c_0 r,
\]

for a positive \( c_0 \), depending only on \( n \) in Problems A, B, and also on \( \lambda_\pm \) for Problem C.

The proof is an application of the mean value theorem and is left as an exercise to the reader.

8.2. **Porosity of the Free Boundary and its Lebesgue measure.**

**Definition 8.7.** We say that the measurable set \( E \subset \mathbb{R}^n \) is porous with porosity constant \( 0 < \delta < 1 \) if every ball \( B = B_r(x) \) contains a smaller ball \( B' = B_{\delta r}(y) \) such that

\[
B_{\delta r}(y) \subset B_r(x) \setminus E.
\]
We say that $E$ is \textit{locally porous} in $D$ if $E \cap K$ is porous (with possibly different porosity constants) for any $K \subset D$.

It is clear that the Lebesgue upper density of a porous set $E$

$$d(x) := \limsup_{r \to 0} \frac{|E \cap B_r(x_0)|}{|B_r(x_0)|} \leq 1 - \delta^n < 1,$$

which implies that $E$ must have Lebesgue measure zero.

\textbf{Proposition 8.8.} \textit{If $E \subset \mathbb{R}^n$ is porous then $|E| = 0$. If $E$ is locally porous in $D$, then $|E \cap D| = 0$.} \hfill \Box

An immediate corollary of the nondegeneracy and the $C^{1,1}$ regularity is the following result.

\textbf{Lemma 8.9 (Porosity of the free boundary).} \textit{Let $u$ be a solution of Problem A, B in an open set $D \subset \mathbb{R}^n$. Then $\Gamma(u)$ is locally porous for any $K \subset D$.}

\textit{If $u$ is solution of Problem C, then $\Gamma'(u) = \Gamma(u) \cap \{\nabla u = 0\}$ is locally porous.}

\textbf{Proof.} For Problems A, B, Let $x_0 \in \Gamma(u)$ and $B_r(x_0) \subset D$. Using the non-degeneracy of the gradient (Corollary 8.6), one can find $x_1 \in \overline{B_{r/2}(x_0)}$ such that

$$|\nabla u(x_1)| \geq \frac{c_0}{2} r.$$

Now, using that $M = \|D^2 u\|_{L^\infty(D)} < \infty$, we will have

$$\inf_{B_{\tilde{r}_r}(x_1)} |\nabla u| \geq \left( \frac{c_0}{2} - M \delta \right) r \geq \frac{c_0}{4} r, \text{ if } \delta = \frac{c_0}{4M}.$$

This implies that

$$B_{\tilde{r}_r}(x_1) \subset B_r(x_0) \cap \Omega(u) \subset B_r(x_0) \setminus \Gamma,$$

where $\tilde{r} = \min\{\delta, 1\}$. This implies the porosity condition is satisfied for any ball centered at $\Gamma(u)$. It is a now simple exercise to show that porosity condition is satisfied for any ball $B \subset D$ and therefore $\Gamma(u)$ is locally porous.

For Problem C, the same argument as above shows that

$$B_{\tilde{r}_r}(x_1) \subset B_r(x_0) \cap \Omega(u) \cup \Gamma''(u) \subset B_r(x_0) \setminus \Gamma'(u),$$

which implies the local porosity of $\Gamma'(u)$.

\textbf{Corollary 8.10 (Lebesgue measure of $\Gamma$).} \textit{Let $u$ be a solution of Problem A, B, and C in $D$. Then $\Gamma(u)$ has Lebesgue measure zero.}

\textbf{Proof.} In case of Problems A, B the statement follows immediately from the local porosity of $\Gamma(u)$ and Proposition 8.8.

In the case of Problem C, we obtain $|\Gamma'(u)| = 0$. On the other hand $\Gamma''(u)$ is locally a $C^{1,\alpha}$ curve and therefore also has a Lebesgue measure zero. Hence, $|\Gamma(u)| = 0$ also in this case. \hfill \Box

We finish this subsection with the following observation.

\textbf{Lemma 8.11 (Density of $\Omega$).} \textit{Let $u$ be a solution of Problem A, B, and C in $D$ and $x_0 \in \Gamma(u)$. Then}

$$|B_r(x_0) \cap \Omega(u)| \geq \beta, \quad (8.7)$$
provided \( B_r(x_0) \subset D \), where \( \beta \) depends only on \( \| D^2 u \|_{L^\infty(D)} \) and \( n \) for Problems A, B and additionally on \( \lambda_\pm \) for Problem C.

**Proof.** The proof of Lemma 8.9 shows that

\[
\frac{|B_r(x_0) \cap \Omega(u)|}{|B_r(x_0)|} \geq \tilde{\delta}^n
\]

in case of Problems A and B and

\[
\frac{|B_r(x_0) \cap \Omega(u)|}{|B_r(x_0)|} = \frac{|B_r(x_0) \cap [\Omega(u) \cup \Gamma''(u)]|}{|B_r(x_0)|} \geq \tilde{\delta}^n
\]

in case of Problem C. This completes the proof. \( \square \)

### 8.3. Hausdorff Measure of the Free Boundary

The porosity if the free boundary not only implies that the its Lebesgue measure is zero but also that it actually has a Hausdorff dimension less than \( n \). A stronger result is as follows.

**Lemma 8.12 (Hausdorff measure of \( \Gamma \)).** Let \( u \) be a \( C^{1,1} \) solution of Problem A, B, or C in an open set \( D \subset \mathbb{R}^n \). Then \( \Gamma(u) \) is a set of finite \((n - 1)\)-dimensional Hausdorff measure locally in \( D \).

**Proof.** Let

\[ v_i = \partial_{x_i} u, \quad E_\epsilon = \{ 0 < |\nabla u| < \epsilon \}. \]

Observe that

\[ c_0 \leq |\Delta u|^2 \leq c_n \sum_{i=1}^n |\nabla v_i|^2 \quad \text{in } \Omega, \]

where \( c_0 = 1 \) in case of Problems A, B and \( c_0 = \min\{\lambda_+^2, \lambda_-^2\} \) for Problem C. Thus, for an arbitrary ball \( B \subset \subset D \) we have

\[ c_0 |B \cap E_\epsilon| \leq c_n \int_{B \cap E_\epsilon} \sum_i |\nabla v_i|^2 dx \leq c_n \sum_i \int_{B \cap \{0 < |v_i| < \epsilon\}} |\nabla v_i|^2 dx. \]

To estimate the right hand side here we apply Lemma 6.2 (from Lecture 6) noticing that \( M_1 = M_2 = 0 \) for Problems A, B, and C. It gives

\[ \int_D |\nabla v_i| \nabla \eta dx \leq 0, \quad i = 1, \ldots, n \]

for any non-negative \( \eta \in C_0^\infty(D) \). These inequalities continue to hold for non-negative \( \eta \in W_0^{1,2}(D) \). Take \( \eta = \psi_t(v_i^\pm)\phi \), with

\[ \psi_t(t) = \begin{cases} 0, & 0 \leq t < \epsilon \\ \epsilon^{-1}t, & 0 \leq t \leq \epsilon \\ 1, & t \geq \epsilon \end{cases} \]

and \( \phi \in C_0^\infty(D) \), \( \phi \geq 0 \). We obtain

\[ \epsilon^{-1} \int_{B \cap \{0 < |v_i| < \epsilon\}} |\nabla v_i|^2 \phi dx \leq \int_D |\nabla v_i||\nabla \phi| dx \leq c_n M \int_D |\nabla \phi| dx \]

In particular, taking \( \phi = 1 \) on \( B \), after summation by \( i \), we arrive at the estimate

(8.8) \[ c_0 |B \cap E_\epsilon| \leq C \epsilon M, \]

where \( C = C(n, B, D) \).
Consider now a covering of $\Gamma \cap B$ by balls $B_i$ of radius $\epsilon$ with centers on $\Gamma \cap B$ and with property that at most $N$ balls may overlap. For Problems A and B, we use (8.7), (8.8) and observe $|\nabla u| \leq M\epsilon$ for in each $B_i$, which gives

$$\sum_i |B_i| \leq \frac{1}{\beta} \sum_i |B_i \cap \Omega| \leq \frac{1}{\beta} \sum_i |B_i \cap E_{M\epsilon}|$$

$$\leq \frac{N}{\beta} |B \cap E_{M\epsilon}| \leq \frac{cNM\epsilon}{c_0\beta}.$$ 

This gives the estimate

$$H^{n-1}(\Gamma(u) \cap B) \leq C(n, M, B, D).$$

For Problem C, the same proof works for $\Gamma'$, and the rest part of the free boundary $\Gamma''$ is smooth. \hfill \Box

Remark 8.13. The estimate (8.8) essentially means

$$|\Omega \cap \{|\nabla u| < \epsilon\}| \leq C\epsilon,$$

which we will use later. In particular, it gives $|\Omega \cap \{|\nabla u| = 0\}| = 0.$