Let $1 < p < \infty$ and denote the norm of the Beurling-Ahlfors transform $B$ on $L^p(C)$ by $\|B\|_{p \to p}$. The estimate $\|B\|_{p \to p} \leq 2(p^* - 1)$, where $p^* = \max\{p, p/(p - 1)\}$, obtained recently by F. Nazarov and A. Volberg [13] is proved here using the martingale techniques of [3] applied to space time Brownian motion.

1. INTRODUCTION

Let $f \in L^p(C)$, $1 < p < \infty$, and define the Beurling-Ahlfors transform by the singular integral operator

$$Bf(z) = -\frac{1}{\pi} \int_{C} \frac{f(\omega)}{\omega - z}^2 dm(\omega),$$

where $dm(\omega)$ is the Lebesgue measure in the complex plane $C$. This operator is the two dimensional analogue of the Hilbert transform and it plays a fundamental role in the study of quasiconformal mappings, partial differential equations and complex analysis. The general theory of singular integrals [14] implies that $B$ is bounded on $L^p(C)$ for any $1 < p < \infty$. We denote its norm by $\|B\|_{p \to p}$. It was conjectured by T. Iwaniec [9] that $\|B\|_{p \to p} = p^* - 1$, where $p^* = \max\{p, p/(p - 1)\}$. A proof of this conjecture will have many important consequences in the theory of quasiconformal mappings and on the regularity of solutions to certain partial differential equations (see [1], [9], [10] and [11]). It is well known that $\|B\|_{p \to p} \geq p^* - 1$. Using some extensions of the inequalities of D. L. Burkholder ([4], [6]) on differential subordination of martingales and a stochastic integral representation of the operator $B$, R. Bañuelos and G. Wang [3] proved that $\|B\|_{p \to p} \leq 4(p^* - 1)$ and that $\|B\|_{p \to p} \leq 2\sqrt{2}(p^* - 1)$ when the operator is restricted to real valued functions. More recently, F. Nazarov and A. Volberg [13] obtained the estimate

$$\|B\|_{p \to p} \leq 2(p^* - 1),$$

improving the previous estimate by a factor of 2. They also proved that when the operator is restricted to real valued functions,
(2) \[ \|B\|_{p \to p} \leq \sqrt{2}(p^* - 1). \]

The purpose of this paper is to show that the same martingale techniques presented in Bañuelos and Wang [3] can be used to produce these improvements. What is needed is to replace the Brownian motion in the arguments of [3] by the space time Brownian motion. The idea to replace the Brownian motion by the space time Brownian motion arose from reading [13] which uses a heat equation version of Lemma 2 on page 87 of [14], applied to an appropriately constructed Bellman functions. It should be mentioned here that the construction of the Bellman function in [13] is also based on Burkholder’s results on differential subordination of martingales applied to Haar functions. In fact, what they use is a special case of the result in [5]. Their arguments, however, do not use any stochastic integration. Nevertheless, it is interesting to note that the key estimate in both [3] and [13], as well as in this paper, is Burkholder’s inequality on the differential subordination of martingales and that at this point, as far as we know, there are no non-martingale proofs of any of these estimates of \( \|B\|_{p \to p} \).

The connection between martingale transforms and singular integrals has been extensively studied in the literature. We refer the reader to [2], [3], [7] and [8] for more applications. The reader interested in other applications of space time Brownian motion to Littlewood–Paley theory should consult P. A. Meyer [12].

2. Space–time Brownian motion, martingale transforms, and the estimates (1) and (2)

By standard density arguments it suffices to prove that for any \( 1 < p < \infty \), \( \|B\varphi\|_p \leq 2(p^* - 1)\|\varphi\|_p \), for all complex valued functions \( \varphi \in C_0^\infty(\mathbb{R}^2) \). Toward this end, fix such a function. The heat kernel for half the Laplacian \( \frac{1}{2} \Delta = \frac{1}{2} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \) in \( \mathbb{R}^2 \) is given by

\[ P_t(z) = \frac{1}{2\pi t} \exp \left( \frac{-|z|^2}{2t} \right), \]

where \( |z| \) is the Euclidean norm of \( z = (x, y) \) in \( \mathbb{R}^2 \), and its Fourier transform (with \( \xi = (\xi_1, \xi_2) \)) is

\[ \hat{P}_t(\xi) = \int_{\mathbb{R}^2} e^{2\pi i z \cdot \xi} P_t(z) \, dz = e^{-2\pi^2 t|\xi|^2}, \]

([14], p. 131). The heat extension of the function \( \varphi \) to the upper half space, \( \mathbb{R}_+^3 = \mathbb{R}^2 \times \mathbb{R}_+ \), is

\[ U_\varphi(z, t) = \int_{\mathbb{R}^2} \varphi(w) P_t(z - w) \, dw. \]
This function solves the heat equation in $\mathbb{R}^3_+$ with boundary values $\varphi$. That is,

$$
\begin{align*}
\frac{\partial U_\varphi(z,t)}{\partial t} &= \frac{1}{2} \Delta U_\varphi(z,t), \quad (z,t) \in \mathbb{R}^3_+ \\
U_\varphi(0,z) &= \varphi(z), \quad z \in \mathbb{R}^2.
\end{align*}
$$

We now follow the construction in [12]. Let $Z_t$ be two dimensional Brownian motion with initial distribution the Lebesgue measure $m$. Fix $T > 0$ and consider the space–time Brownian motion $B_t = (Z_t, T - t), \ t \in [0, T]$. This process starts on the hyperplane $\mathbb{R}^2 \times \{T\}$ with initial distribution $m \otimes \delta_T$. Let $P^T$ denote the “probability” measure associated with this process, and denote by $E^T$ the corresponding expectation. Fubini’s theorem implies that for all functions $\varphi$ as above

$$
E^T[\varphi(Z_T)] = E^T[U_\varphi(B_T)] = \int_{\mathbb{R}^2} E_z[U_\varphi(Z_T, 0)] \, dz
$$

$$
= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \varphi(z) P_T(z - w) \, dz \, dw
$$

$$
= \int_{\mathbb{R}^2} \varphi(z) \, dz.
$$

It is well known that $U_\varphi(B_t) = U_\varphi(Z_t, T - t), \ t < T$, is a martingale and Itô’s formula gives that

$$
\varphi(Z_T) = U_\varphi(B_T) = U_\varphi(B_0) + \int_0^T \nabla_z U_\varphi(B_t) \cdot dZ_t
$$

where

$$
\nabla_z U_\varphi(\cdot) = \left(\frac{\partial U_\varphi}{\partial x}(\cdot), \frac{\partial U_\varphi}{\partial y}(\cdot)\right).
$$

We define, for any $2 \times 2$ matrix $A$, the martingale transform of $U_\varphi(B_t)$ by

$$
A \ast U_\varphi = \int_0^T \left[ A \nabla_z U_\varphi(B_t) \right] \cdot dZ_t,
$$

and its projection in $\mathbb{R}^2$ by

$$
S^T_A \varphi(x) = E^T\left[ A \ast U_\varphi \mid B_T = (x, 0) \right].
$$

Burkholder’s result on differential subordination of martingales, as presented in Theorem 4.2 in [3], gives that for all $\varphi \in C^\infty_0(\mathbb{R}^2)$

$$
\|S^T_A \varphi\|_p \leq (p^*-1) \|A\| \left(\int_0^T \|\nabla_z U_\varphi(B_s) \cdot dZ_s\|_p\right)
$$

$$
= (p^*-1) \|A\| \left(\left(E^T\left[\varphi(B_T) - U(B_0)\right]\right)^p\right)^{1/p},
$$

where $\|A\| = \sup\{\|A(z, \omega)\|_{C^2} : z, \omega \in C, \|(z, \omega)\|_{C^2} \leq 1\}$.

The estimate for the norm of the Beurling-Ahlfors operator will follow from this. It is easy to show that for all $1 < p < \infty$, $\|\varphi(B_T) - U_\varphi(B_0)\|_p \leq 2\|\varphi\|_p$. This simple estimate, however, is not good enough for our purpose. The following proposition provides the improvement we need.
Proposition 2.1. For all \( \varphi \in C_0^\infty(\mathbb{R}^2) \) and all \( 2 \leq p < \infty \),

\[
\lim_{T \to \infty} E^T \left[ |\varphi(Z_T) - U_\varphi(B_0)|^p \right] \leq \lim_{T \to \infty} E^T \left[ |\varphi(Z_T)|^p \right] = \|\varphi\|_p^p.
\]

Proof: Fix \( z \in \mathbb{C} \). Let \( K \) be the support of \( \varphi \) and let \( Z_t \) be a two dimensional Brownian motion starting at \( z \). If \( p \geq 2 \) is a positive integer we have that

\[
E_z \left[ |\varphi(Z_T) - U_\varphi(Z_0, T)|^p \right] \leq E_z \left[ \sum_{k=0}^{p} \left( \begin{array}{c} p \\ k \end{array} \right) |U_\varphi(z, T)|^{p-k} |\varphi(Z_T)|^k \right] 
\]

(6) \( E_z \left[ |\varphi(Z_T)|^p \right] + \sum_{k=0}^{p-1} \left( \begin{array}{c} p \\ k \end{array} \right) \left( E_z \left[ |\varphi(Z_T)|^k \right] \right)^{p-k} E_z \left[ |\varphi(Z_T)|^k \right]. \)

If \( 0 < k < p \) we obtain from Hölder’s inequality that

\[
\left( E_z \left[ |\varphi(Z_T)|^k \right] \right)^{p-k} \leq E_z \left[ |\varphi(Z_T)|^{p-k} \right] \left( P_z \left[ Z_T \in K \right] \right)^{p-k-1},
\]

and

\[
E_z \left[ |\varphi(Z_T)|^k \right] \leq \left( P_z \left[ Z_T \in K \right] \right)^{1-\left( k/p \right)} \left( E_z \left[ |\varphi(Z_T)|^p \right] \right)^{k/p}.
\]

Therefore

\[
\left( E_z \left[ |\varphi(Z_T)|^p \right] \right)^{p-k} E_z \left[ |\varphi(Z_T)|^k \right] \leq E_z \left[ |\varphi(Z_T)|^{p-k} \right] \left( P_z \left[ Z_T \in K \right] \right)^{p-k-1} E_z \left[ |\varphi(Z_T)|^k \right] \leq \left( E_z \left[ |\varphi(Z_T)|^p \right] \right)^{\left[ k/p \right] + \left[ (p-k)/p \right]} \left( P_z \left[ Z_T \in K \right] \right)^{p-k-1 + 1 - \left[ k/p \right] + 1 - \left[ (p-k)/p \right]}.
\]

Combining this inequality with (6) we have that

\[
E_z \left[ |\varphi(Z_T) - U_\varphi(Z_0, T)|^p \right] \leq E_z \left[ |\varphi(Z_T)|^p \right] + \sum_{k=0}^{p-1} \left( \begin{array}{c} p \\ k \end{array} \right) E_z \left[ |\varphi(Z_T)|^p \right] \left( P_z \left[ Z_T \in K \right] \right)^{p-k}.
\]

We deduce that there exists a constant \( C(p) \), depending only on \( p \), such that for all positive integer \( p \geq 2 \)

\[
E_z \left[ |\varphi(Z_T) - U_\varphi(Z_0, T)|^p \right] \leq E_z \left[ |\varphi(Z_T)|^p \right] \left[ 1 + C(p) P^z \left[ Z_T \in K \right] \right] \leq E_z \left[ |\varphi(Z_T)|^p \right] \left[ 1 + C(p) \frac{m(K)}{2\pi T} \right] .
\]

Consider now the space \( \mathcal{E} = L^p(K, \mu_{T,z}) \), where

\[
d\mu_{T,z} = P_T(w-z)dw,
\]
and the operator $H$ on $E$ given by

$$H(\phi) = \phi - \int_K \phi(w) d\mu_{T,z}(w).$$

Then the last inequality implies that, for all $\phi \in E$ and all positive integer $p \geq 2$, we have that

$$\|H(\phi)\|_E = (E_z|\varphi(Z_T) - U\varphi(Z_0, T)|^p)^{1/p} \leq \left[1 + \frac{C(p) m(K)}{2\pi T}\right]^{1/p} \|\phi\|_E.$$

Thus the Riesz-Thorin Interpolation theorem implies that for all $p \geq 2$, and $\varphi \in C_0^\infty(\mathbb{R}^2)$

$$E_z\left[|\varphi(Z_T) - U\varphi(Z_0, T)|^p\right] \leq E_z\left[|\varphi(Z_T)|^p\right] \left[1 + \frac{C'(p) m(K)}{2\pi T}\right]$$

where the constant $C'(p)$ can be taken to be $\max\{C([p]), C([p]+1)\}$ and $[p]$ the largest integer smaller than $p$. Integrating in $z$ over $\mathbb{R}^2$ we have that

$$E^T\left[|\varphi(Z_T) - U\varphi(B_0)|^p\right] \leq \left[1 + \frac{C'(p) m(K)}{2\pi T}\right] E^T\left[|\varphi(Z_T)|^p\right].$$

We conclude that

$$\lim_{T \to \infty} E^T\left[|\varphi(Z_T) - U\varphi(B_0)|^p\right] \leq \lim_{T \to \infty} E^T\left[|\varphi(Z_T)|^p\right] = \|\varphi\|_p^p,$$

proving the Proposition. •

Next, let $R_1$ and $R_2$ denote the Riesz transforms in $\mathbb{R}^2$ with Fourier multipliers

$$m_1(\xi) = \frac{i\xi_1}{|\xi|}, \quad m_2(\xi) = \frac{i\xi_2}{|\xi|},$$

respectively. We recall that the Fourier multiplier of $B$ is given by

$$m(\xi) = \frac{\xi}{|\xi|}.$$

Thus we have the following representation of $B$ in terms of the Riesz transforms in $\mathbb{R}^2$:

(7) \hspace{1cm} B = R_2^2 - R_1^2 + 2iR_2R_1.

**Proposition 2.2.** Let $i, j \in \{1, 2\}$ and $A^{i,j} = (a_{r,s}^{i,j})$ be the $2 \times 2$ real matrix defined by

$$a_{r,s}^{i,j} = -1 \text{ and } a_{r,s}^{i,j} = 0 \text{ if } r \neq i \text{ or } s \neq j.$$

Then for all $\varphi \in C_0^\infty(\mathbb{R}^2)$

(8) \hspace{1cm} \lim_{T \to \infty} \int_{\mathbb{R}^2} g(z)S_{A^{i,j}}^T \varphi(z) dz = \int_{\mathbb{R}^2} g(z)R_iR_j\varphi(z) dz,

for any $g \in L^q(\mathbb{R}^2)$, $1 < q < \infty$. 
Proof: We will only prove the case \( i = j = 1 \), the other cases follow a similar argument. To make the notation a little simpler assume, by splitting into real and imaginary parts, that \( \varphi \) is real valued. Let \( \psi \in C_0^\infty(\mathbb{R}^2) \), which we also assume to be real valued. By (3),

\[
\int_{\mathbb{R}^2} \psi(z) S_{A_{1.1}}(z)dz = -E_T \left[ \psi(B_T) \int_0^T \frac{\partial \varphi}{\partial x}(B_t) dZ_t^1 \right]
\]

(9)

\[
= -E_T \left[ \psi(B_0) \int_0^T \frac{\partial \varphi}{\partial x}(B_t) dZ_t^1 \right]
- E_T \left[ \int_0^T \nabla \cdot U \psi(B_s) \cdot dZ_t \int_0^T \frac{\partial \varphi}{\partial x}(B_t) dZ_t^1 \right].
\]

We claim that

\[
\lim_{T \to \infty} E_T \left[ \psi(B_0) \int_0^T \frac{\partial \varphi}{\partial x}(B_t) dZ_t^1 \right] = 0.
\]

(10)

Applying the Cauchy-Schwarz inequality and the \( L^2 \)-isometry of stochastic integrals, we get

\[
E_T \left| U \psi(B_0) \int_0^T \frac{\partial \varphi}{\partial x}(B_t) dZ_t^1 \right| \leq \left( E_T \left| U \psi(B_0) \right|^2 \right)^{1/2} \left( E_T \left| \int_0^T \frac{\partial \varphi}{\partial x}(B_t) dZ_t^1 \right|^2 \right)^{1/2} \leq \left( E_T \left| U \psi(B_0) \right|^2 \right)^{1/2} \left( E_T \left| \int_0^T \frac{\partial \varphi}{\partial x}(B_t) dt \right|^2 \right)^{1/2} \leq \left( E_T \left| U \psi(B_0) \right|^2 \right)^{1/2} \left( E_T \left| \nabla \varphi(B_t) \right|^2 \right)^{1/2} = \left( E_T \left| U \psi(B_0) \right|^2 \right)^{1/2} \left( E_T \left| \varphi(Z_T) - U \phi(B_0) \right|^2 \right)^{1/2}
\]

By Proposition 2.1,

\[
\lim_{T \to \infty} E_T \left| U \psi(B_0) \right| = \left| \psi \right|_2^2.
\]

On the other hand, since \( |U \psi(z, T)| \leq \frac{\| \psi \|_1}{2\pi T} \), uniformly in \( z \) we have,

\[
E_T \left| U \psi(B_0) \right|^2 = \int_{\mathbb{R}^2} E_T |U \psi(B_0)|^2 dz
= \int_{\mathbb{R}^2} |U \psi(z, T)|^2 dz
\leq \frac{\| \psi \|_1}{2\pi T} \int_{\mathbb{R}^2} |U \psi(z, T)| dz
\leq \frac{\| \psi \|_1^2}{2\pi T},
\]

which goes to zero as \( T \to \infty \). This proves the claim.
Returning to (9) the definition of $E^T$ and Parseval’s formula imply that
\[
\begin{align*}
-E^T \left[ \int_0^T \frac{\partial U_\varphi}{\partial x}(B_t) dZ_t \int_0^T \nabla z U_\psi(B_t) \cdot dZ_t \right] & = -E^T \left[ \int_0^T \frac{\partial U_\varphi}{\partial x}(B_t) \frac{\partial U_\psi}{\partial x}(B_t) dt \right] \\
& = -\int_0^T \int_{\mathbb{R}^2} \frac{\partial U_\varphi}{\partial x}(z, T-t) \frac{\partial U_\psi}{\partial x}(z, T-t) P_t(z-w) dz dw dt \\
& = -\int_0^T \int_{\mathbb{R}^2} \frac{\partial U_\varphi}{\partial x}(\xi, T-t) \frac{\partial U_\psi}{\partial x}(\xi, T-t) \xi dt \\
& = -\int_0^T \int_{\mathbb{R}^2} 4\pi^2 \xi_1^2 e^{-4\pi^2(T-t)|\xi|^2} \hat{\varphi}(\xi) \overline{\hat{\psi}(\xi)} d\xi dt.
\end{align*}
\]
A simple change of variables, the dominated convergence theorem and one more use of Parseval’s formula give
\[
\begin{align*}
-\lim_{T \to \infty} & \int_0^T \int_{\mathbb{R}^2} 4\pi^2 \xi_1^2 e^{-4\pi^2(T-t)|\xi|^2} \hat{\varphi}(\xi) \overline{\hat{\psi}(\xi)} d\xi dt \\
& = -\left( \int_0^\infty \int_{\mathbb{R}^2} 4\pi^2 \xi_1^2 e^{-4\pi^2t|\xi|^2} \hat{\varphi}(\xi) \overline{\hat{\psi}(\xi)} d\xi dt \right) \\
& = -\int_{\mathbb{R}^2} \frac{\xi_1^2}{|\xi|^2} \hat{\varphi}(\xi) \overline{\hat{\psi}(\xi)} d\xi \\
& = \int_{\mathbb{R}^2} \psi(z) R_1^2 \varphi(z) dz,
\end{align*}
\]
Thus,
\[
\lim_{T \to \infty} \int_{\mathbb{R}^2} \psi(z) S^T_{A_1,1} \varphi(z) dz = \int_{\mathbb{R}^2} \psi(z) R_1^2 \varphi(z) dz,
\]
for all $\psi \in C_0^\infty(\mathbb{R}^2)$. By (5) and the fact that $\|\varphi(B_T) - U(B_0)\|_p \leq 2\|\varphi\|_p$ for any $1 < p < \infty$, we see that $\|S^T_{A_1,1} \varphi(z)\|_p \leq C_p \|\varphi\|_p$ with a constant independent of $T$. Of course, we also know that $\|R_1^2 \varphi\|_p \leq C_p \|\varphi\|_p$, for any $1 < p < \infty$. From this, (11), and the density of $C_0^\infty(\mathbb{R}^2)$ in $L^p(\mathbb{R}^2)$, $1 < p < \infty$ the proposition follows.

**Theorem 2.1.** Let $f \in L^p(\mathbb{C})$, $1 < p < \infty$. Then
\[
\|Bf\|_p \leq 2(p^*-1)\|f\|_p.
\]
If in addition, $f : \mathbb{C} \to \mathbb{R}$, then
\[
\|Bf\|_p \leq \sqrt{2}(p^*-1)\|f\|_p
\]
Proof: Consider the matrix
\[
\mathcal{B} = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}
\]
and observe that \( \|B\| = 2 \). Duality and Propositions 2.1 and 2.2, together with (4), (5) and (7), give that for any \( p \geq 2 \) and any \( \varphi \in C_0^{\infty}(\mathbb{R}^2) \),

\[
\|B\varphi\|_p \leq 2(p^*-1)\|\varphi\|_p.
\]

The density of \( C_0^{\infty}(\mathbb{R}^2) \) in \( L^p(\mathbb{C}) \) implies (i) for all \( p \geq 2 \), and the case \( 1 < p < 2 \) follows by duality. Finally, (ii) follows from the fact that the norm of \( B \) acting on real vectors is \( \sqrt{2} \).

The above results can be derived for the Riesz transform in \( \mathbb{R}^n \) for any \( n \geq 2 \) and we obtained the following improvement of the first result in Theorem 4 of [3].

**Theorem 2.2.** Let \( f \in L^p(\mathbb{R}^n) \), \( 1 < p < \infty \). Then

\[
\left\| \sum_{j=1}^n a_j R_j^2 f \right\|_p \leq (p^*-1)\|f\|_p, \quad a_j \in \{-1, 0, 1\}
\]

and

\[
\|R_j R_k f\|_p \leq \frac{1}{2}(p^*-1)\|f\|_p, \quad j \neq k.
\]

Furthermore, if \( n \) is even, say \( n = 2m \), then

\[
\left\| \sum_{j=1}^m a_{2j} R_{2j-1} R_{2j} f \right\|_p \leq \frac{1}{2}(p^*-1)\|f\|_p, \quad a_j \in \{-1, 0, 1\}.
\]

This result shows that in particular, \( \|(R_2^2 - R_1^2) f\|_p \leq (p^*-1)\|f\|_p \) and that \( \|2R_1 R_2 f\|_p \leq (p^*-1)\|f\|_p \). Hence, as in the case of the results in [3], the argument does not really treat the Beurling–Ahlfors operator as a single entity. In particular, the argument here does not provide the right constant for \( p = 2 \).

We end with some remarks on the difference between the martingale representation of \( B \) presented here and the one presented in [2], [3] and [8]. In those papers the martingale transform is given in terms of the harmonic extension of \( \varphi \) to \( \mathbb{R}^3_+ \) instead of the heat extension used in this paper. The question arises if one can use other processes to obtain different martingale representations of \( B \) and perhaps improve the estimates on \( \|B\|_{p\to p} \). In particular, it is natural to try to extend \( \varphi \) using the densities of other stable processes or to try to use a Bessel process for the vertical component. We were not able to improve on the constants any further with that strategy.

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