MA 181 Exam 1 Solutions

1. a) \( L = \int_{0}^{\pi} \sqrt{1 + [2 \cos 2x]^2} \, dx \)

b) \( A = \int_{0}^{\pi} 2\pi \sin(2x) \sqrt{1 + [2 \cos 2x]^2} \, dx \)

2. a) \( A = \frac{1}{2} | \mathbf{AB} \times \mathbf{AC} | \)

    \[ = \frac{1}{2} \left| \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 0 & 1 \\ 3 & 2 & 0 \end{bmatrix} \right| \]

    \[ = \frac{1}{2} \left| \mathbf{2i} + \mathbf{2j} - 3\mathbf{k} \right| \]

    \[ = \frac{1}{2} \sqrt{4 + 4 + 9} = \frac{\sqrt{17}}{2} \]

b) \( \mathbf{n} = \frac{\mathbf{AB} \times \mathbf{AC}}{\left| \mathbf{AB} \times \mathbf{AC} \right|} = \frac{\mathbf{2i} + \mathbf{2j} - 3\mathbf{k}}{\sqrt{17}} \)
c) \[ \cos \theta = \frac{\vec{AC} \cdot \vec{AB}}{|\vec{AC}| |\vec{AB}|} = \frac{(\hat{i} + 2\hat{j} + 2\hat{k}) \cdot (-\hat{i} + \hat{j})}{\sqrt{1+4+4} \cdot \sqrt{1+1}} = \frac{1 \cdot (-1) + 2 \cdot 1 + 2 \cdot 0}{\sqrt{9} \sqrt{2}} = \frac{1}{3\sqrt{2}} \]

d) \[
\text{Proj}_{\vec{AB}} \vec{AC} = \frac{\vec{AC} \cdot \vec{AB}}{|\vec{AB}|^2} \vec{AB} = \frac{1}{(\sqrt{1+1})^2}(-\hat{i} + \hat{j}) = -\frac{1}{2}\hat{i} + \frac{1}{2}\hat{j}
\]

3. \[
\begin{align*}
  f'(t) &= \int \sin(3t) \, dt = -\frac{1}{3} \cos 3t + C \\
  f(t) &= \int -\frac{1}{3} \cos 3t + C \, dt \\
  &= -\frac{1}{9} \sin 3t + Ct + C_2 \quad \text{want}
\end{align*}
\]

\[
\begin{align*}
  f(0) &= -\frac{1}{9} \sin 0 + C \cdot 0 + C_2 = 5 \\
  \text{so } C_2 &= 5 \text{ and we get}
\end{align*}
\]
\[ f(t) = -\frac{1}{9} \sin(3t) + ct + 5 \]

where \( C \) is an arbitrary constant.

4. Cylindrical shells:

\[ V = \int_{0}^{\pi/4} 2\pi x (\cos x - \sin x) \, dx \]

Washers:

\[ V = \int_{0}^{\sqrt{2}/2} \pi \left( \sin^{-1} y \right)^2 \, dy + \int_{\sqrt{2}/2}^{1} \pi \left( \cos^{-1} y \right)^2 \, dy \]

5. Let \( u = 3x + 2 \). Then

\[ du = 3\, dx \quad \text{and} \quad dx = \frac{1}{3} \, du \]

Also \( x = \frac{1}{3}(u-2) \). So

\[ \int x \left( 3x + 2 \right)^{100} \, dx = \]
\[
\int \frac{1}{3} (u-2) \cdot u^{100} \left( \frac{1}{3} \, du \right) \\
= \frac{1}{9} \int u^{101} - 2u^{100} \, du \\
= \frac{1}{9} \left[ \frac{1}{102} u^{102} - \frac{2}{101} u^{101} \right] + C \\
= \frac{1}{9} \left[ \frac{1}{102} (3x+2)^{102} - \frac{2}{101} (3x+2)^{101} \right] + C
\]

6. \[
V = \int_1^1 \pi \left( x^2 - c \right)^2 \, dx \\
= \int_1^1 \pi \left( x^4 - 2x^2c + c^2 \right) \, dx \\
= \pi \left[ \frac{1}{5} x^5 - \frac{2}{3} x^3 c + c^2 x \right]_{-1}^1 \\
= \pi \left[ \frac{2}{5} - \frac{4}{3} c + 2c^2 \right]
\]
\[ V(c) = \frac{2}{\pi} \left( 2c^2 - \frac{4}{3}c + \frac{2}{5} \right) \]

\[ V'(c) = \frac{2}{\pi} \left( 4c - \frac{4}{3} \right) \]

\[ V'(c) = 0 \quad \text{when} \quad c = \frac{1}{3}. \]

\[ V(0) = \frac{2}{5} \pi \]

\[ V \left( \frac{1}{3} \right) = \frac{2}{\pi} \left( \frac{\frac{2}{3}}{\frac{4}{3}} - \frac{4}{9} + \frac{2}{5} \right) = \left( \frac{\frac{2}{5} - \frac{2}{9} \pi} \right) \]

\[ V(1) = \frac{2}{\pi} \left( 2 - \frac{4}{3} + \frac{2}{5} \right) = \left( \frac{2}{5} + \frac{1}{3} \right) \pi \]

Minimum is at \( c = \frac{1}{3} \).