Name:

Signature:

Student ID Number:

**Directions:** Work on as many of the problems below as you can. Start by working in the space below the problem; if this is not sufficient use the back of another page and clearly mark that the problem is continued elsewhere. Do not take the exam apart. Be sure to show all work.

There are five problems. The points allotted to each problem are given in parenthesis. The total score is 100.

NO CALCULATORS, BOOKS, OR PAPERS ARE ALLOWED. Use the back of the test pages for scrap paper.

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Problem 1: Consider the matrix

\[ A = \begin{bmatrix}
8 & 2 & 6 \\
16 & 6 & 32 \\
4 & 0 & -7
\end{bmatrix} \]

a) Solve \( Ax = 0 \) and write down a basis for the null space. 
\((10\ points)\)

b) What is the nullity of \( A \)? \((3\ points)\)

c) What is the rank of \( A \)? \((3\ points)\)

d) Does the inverse of \( A \) exist? Why or why not? \((4\ points)\)
Problem 2: Consider the matrix

\[ A = \begin{bmatrix} 4 & 2 \\ -4 & -2 \end{bmatrix} \]

a) Find the eigenvalues and corresponding eigenvectors of \( A \).

\( (8 \text{ points}) \)

b) Diagonalize \( A \), that is find \( P \) and \( D \) such that \( D \) is diagonal, and \( P^{-1}AP = D \).

\( (7 \text{ points}) \)

c) Calculate \( P^{-1} \).

\( (5 \text{ points}) \)
Problem 3: Consider the matrix

\[ A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \]

a) Is the matrix \( A \) orthogonal? Give a reason for your answer. 
(8 points)

b) What do you know about the eigenvalues of \( A \) from you answer in part a)? 
(4 points)

c) Calculate the eigenvalues and eigenvectors of \( A \). 
(8 points)
Problem 4: Consider the second order differential equation
\[ y'' + 25y = 0. \]

a) Convert the equation to a system of first-order differential equations. (5 points)

b) Calculate the general solution of the system found in part a). In case your answer is complex, write down a real general solution. (7 points)

c) What kind of critical point is the origin? Is it stable or unstable? (4 points)

d) Draw a qualitative phase portrait for the vicinity of the origin. (4 points)
**Problem 5:** Consider the system of equations

\[
\begin{align*}
\frac{dy_1}{dt} &= -y_1 + y_2 + y_1 y_2 \\
\frac{dy_2}{dt} &= -y_1 - y_2
\end{align*}
\]

a) Find the critical points of the system. (*5 points*)

b) In the same order as they are listed in part a), tell if the critical points are nodes, saddle points, centers or spiral points. (*8 points*)

c) In the same order as they are listed in part a), tell if the critical points are stable, stable and attractive or unstable. Give a reason for your answer in each case. (*7 points*)