1. Suppose that $f$ is a non-vanishing analytic function on the complex plane minus the origin. Let $\gamma$ denote the curve given by $z(t) = e^{it}$ where $0 \leq t \leq 2\pi$. Suppose that
\[
\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} \, dz
\]
is divisible by 3. Prove that $f$ has an analytic cube root on $\mathbb{C} - \{0\}$.

2. Suppose that $\{a_k\}_{k=1}^N$ is a finite sequence of distinct complex numbers and that $f$ is analytic on $\mathbb{C} - \{a_k : k = 1, 2, \ldots, N\}$. Prove that there exist constants $c_j$, $j = 1, 2, \ldots, N$, such that
\[
f(z) = \sum_{k=1}^{N} \frac{c_k}{z - a_k}
\]
has an analytic antiderivative on $\mathbb{C} - \{a_k : k = 1, 2, \ldots, N\}$.

3. How many zeroes does the polynomial
\[z^{1998} + z + 2001\]
have in the first quadrant? Explain your answer.

4. Suppose that $u$ is continuous on $\overline{D_1(0)}$ and that
\[
(*) \quad u(z) = \frac{1}{2\pi} \int_{0}^{2\pi} u(z + (1 - |z|)e^{i\theta}) \, d\theta
\]
for each $z \in D_1(0)$. (This equality means that $u$ is only known to satisfy the averaging property on circles like the one pictured below.) Prove that $u$ is harmonic in $D_1(0)$.

Note: $(*)$ means that $u(z)$ is equal to the average of $u$ over the internally tangent circle centered at $z$. 