A square summable power series $f(z)$ converges in the unit disk and represents a function $f(w)$ of $w$ in the unit disk whose value at $w$ is a scalar product

$$f(w) = \langle f(z), (1 - w^{-1}z)^{-1} \rangle_{C(z)}$$

with an element

$$(1 - w^{-1}z)^{-1} = \sum (w^n)^{-1} z^n$$

of the Hilbert space. Since the power series is uniquely determined by the function, the power series is identified with the function which it represents. The represented function is continuous in the unit disk. It is also differentiable at $w$ when $w$ is the unit disk. The difference quotient

$$\frac{f(z) - f(w)}{z - w}$$

is represented by a square summable power series.

A fundamental theorem of analytic function theory states that a function which is differentiable in the unit disk is represented by a power series. If a function $W(z)$ of $z$ in the unit disk is differentiable and bounded by one, then $W(z)$ is represented by a square summable power series. Proofs of the representation theorem relate geometric properties of functions to their analytic equivalents.

The maximum principle states that a differentiable function $f(z)$ of $z$ in the unit disk, which has a continuous extension to the closure of the unit disk and which is bounded by one on the unit circle is bounded by one in the disk. A contradiction results from the assumption that such a function has values which lie outside of the closure of the unit disk.

Since the function maps the closure of the unit disk onto a compact subset of the complex plane, the complex complement of the set of values is a nonempty open set whose boundary is not contained in the closure of the unit disk. Elements of the unit disk exist which are mapped into the part of the boundary which lies outside of the closed disk. The derivative is easily seen to be zero at such elements of the disk. Such elements $a$ and $b$ of the unit disk are considered equivalent if no disjoint open subsets $A$ and $B$ of the unit disk exist such that $a$ belongs to $A$, such that $b$ belongs to $B$, and such that the complement in the disk of the union of $A$ and $B$ is mapped into the closure of the disk. An equivalence relation has been defined on such elements of the disk. Equivalent elements can be reached from each other by a chain in the equivalence class. Since the derivative vanishes on the chain, the function remains constant on the equivalence class. A contradiction is obtained.
since the function maps the unit disk onto a compact subset of the complex plane whose boundary is contained in the closure of the disk.

An application of the maximum principle is made to a function $W(z)$ of $z$ in the unit disk which is differentiable and bounded by one. If $W(w)$ belongs to the disk for some $w$ in the disk, then the function

$$\frac{W(z) - W(w)}{1 - W(z)W(w)^{-}}$$

of $z$ in the disk is differentiable and bounded by one. The function $W(z)$ of $z$ maps the unit disk into itself if it is not a constant of absolute value one.

These properties of a function $W(z)$ of $z$ in the unit disk, which are differentiable and bounded by one in the disk, are sufficient [11] for the construction of a Hilbert space $\mathcal{H}(W)$ whose elements are differentiable functions in the disk. The space contains the function

$$\frac{1 - W(z)W(w)^{-}}{1 - zw^{-}}$$

of $z$, when $w$ is in the disk, as reproducing kernel function for function values at $w$. The identity

$$f(w) = \langle f(z), [1 - W(z)W(w)^{-}]/(1 - zw^{-}) \rangle_{\mathcal{H}(W)}$$

holds for every element $f(z)$ of the space. The elements of the space are continuous functions in the disk. The difference quotient

$$\frac{f(z) - f(w)}{z - w}$$

belongs to the space as a function of $z$ when $w$ is in the space. The elements of the space are represented by square summable power series. The space $\mathcal{H}(W)$ is contained contractively in $C(z)$ when an element of the space is identified with its representing power series. Multiplication by $W(z)$ is a contractive transformation of the space $C(z)$ into itself.

A power series is treated as a Laurent series which has zero coefficients for negative powers of $z$. The space of square summable Laurent series is the Hilbert space $\text{ext } C(z)$ of series

$$\sum a_n z^n$$

defined with summation is over all integers $n$ with a finite sum

$$\|f(z)\|_{\text{ext } C(z)}^2 = \sum a_n^{-} a_n$$

The space $C(z)$ of square summable power series is contained isometrically in the space $\text{ext } C(z)$ of square summable Laurent series. An isometric transformation of $\text{ext } C(z)$ onto itself, which maps $C(z)$ onto its orthogonal complement, is defined by taking $f(z)$ into $z^{-1} f(z^{-1})$. The transformation is its own inverse.
Multiplication transformations are defined in the space of square summable power series by power series. The conjugate of a power series

\[ W(z) = \sum W_n z^n \]

is the power series

\[ W^*(z) = \sum W_n^* z^n \]

whose coefficients are complex conjugate numbers. If \( f(z) \) is a power series,

\[ g(z) = W(z)f(z) \]

is the power series obtained by Cauchy convolution of coefficients. Multiplication by \( W(z) \) in \( C(z) \) is the transformation which takes \( f(z) \) into \( g(z) \) when \( f(z) \) and \( g(z) \) belongs to \( C(z) \). If multiplication by \( W(z) \) is densely defined as a transformation in \( C(z) \), then the adjoint is a transformation whose domain contains the polynomial elements of \( C(z) \). The adjoint transformation maps a polynomial element \( f(z) \) of \( C(z) \) into the polynomial element \( g(z) \) of \( C(z) \) such that

\[ z^{-1}g(z^{-1}) - W^*(z)z^{-1}f(z^{-1}) \]

is a power series. Multiplication by \( W(z) \) in \( C(z) \) is then the adjoint of its adjoint restricted to polynomial elements of \( C(z) \).

A Krein space \( \mathcal{H}(W) \), whose elements are power series, will be constructed from a given power series \( W(z) \) when multiplication by \( W(z) \) is a densely defined transformation in \( C(z) \). The space contains

\[ f(z) - W(z)g(z) \]

whenever \( f(z) \) and \( g(z) \) are elements of \( C(z) \) such that the adjoint of multiplication by \( W(z) \) in \( C(z) \) takes \( f(z) \) into \( g(z) \) and such that \( g(z) \) is in the domain of multiplication by \( W(z) \) in \( C(z) \). The identity

\[ \langle h(z), f(z) - W(z)g(z) \rangle_{\mathcal{H}(W)} = \langle h(z), f(z) \rangle_{C(z)} \]

then holds for every element \( h(z) \) of the space \( \mathcal{H}(W) \) which belongs to \( C(z) \). The series \([f(z) - f(0)]/z\) belongs to the space \( \mathcal{H}(W) \) whenever \( f(z) \) belongs to the space. The Krein space \( \mathcal{H}(W') \) associated with the power series

\[ W'(z) = zW(z) \]

is the set of power series \( f(z) \) with vector coefficients such that \([f(z) - f(0)]/z\) belongs to the space \( \mathcal{H}(W) \). The identify for difference quotients

\[ \langle [f(z) - f(0)]/z, [f(z) - f(0)]/z \rangle_{\mathcal{H}(W)} = \langle f(z), f(z) \rangle_{\mathcal{H}(W')} - f(0)^-f(0) \]

is then satisfied. The resulting properties of the space \( \mathcal{H}(W) \) create [4] a canonical coisometric linear system with transfer function \( W(z) \). The space \( \mathcal{H}(W) \) is the state space of
the linear system. The main transformation, which maps the state space into itself, takes $f(z)$ into
\[ [f(z) - f(0)]/z. \]
The input transformation, which maps the space of complex numbers into the state space, takes $c$ into
\[ |W(z) - W(0)|c/z. \]
The output transformation, which maps the state space into the space of complex numbers, takes $f(z)$ into $f(0)$. The external operator, which maps the space of complex numbers into itself, takes $c$ into
\[ W(0)c. \]

A matrix of continuous linear transformations has been constructed which maps the Cartesian product of the state space and the space of complex numbers continuously into itself. The coisometric property of the linear system states that the matrix has an isometric adjoint.

A Krein space is a vector space with scalar product which is the orthogonal sum of a Hilbert space and the anti-space of a Hilbert space. A Krein space is characterized as a vector space with scalar product which is self-dual for a norm topology.

**Theorem 1.** A vector space with scalar product is a Krein space if it admits a norm which satisfies the convexity identity
\[ \|(1 - t)a + tb\|^2 + t(1 - t)\|b - a\|^2 = (1 - t)\|a\|^2 + t\|b\|^2 \]
for all elements $a$ and $b$ of the space when $0 < t < 1$ and if the linear functionals on the space which are continuous for the metric topology defined by the norm are the linear functionals which are continuous for the weak topology induced by duality of the space with itself.

**Proof of Theorem 1.** Norms on the space are considered which satisfy the hypotheses of the theorem. The hypotheses imply that the space is complete in the metric topology defined by any such norm. If a norm $\|c\|_+$ is given for elements $c$ of the space, a dual norm $\|c\|_-$ for elements $c$ of the space is defined by the least upper bound
\[ \|a\|_- = \sup |\langle a, b \rangle| \]
taken over the elements $b$ of the space such that $\|b\|_+ < 1$.

The least upper bound is finite since every linear functional which is continuous for the weak topology induced by self-duality is assumed continuous for the metric topology. Since every linear functional which is continuous for the metric topology is continuous for the
factorization in Krein spaces

weak topology induced by self-duality, the set of such elements \( b \) is a disk for the weak topology induced by self-duality. The set of elements \( a \) of the space such that

\[
\|a\|_- \leq 1
\]

is compact in the weak topology induced by self-duality. The set of elements \( a \) of the space such that

\[
\|a\|_- < 1
\]

is open for the metric topology induced by the plus norm. Since the set is a disk for the weak topology induced by self-duality, the set of elements \( b \) of the space such that

\[
\|b\|_+ \leq 1
\]

is compact in the weak topology induced by self-duality.

The convexity identity

\[
\|(1 - t)a + t b\|_+^2 + t(1 - t)\|b - a\|_+^2 = (1 - t)\|a\|_+^2 + t\|b\|_+^2
\]

holds by hypothesis for all elements \( a \) and \( b \) of the space when \( 0 < t < 1 \). It will be shown that the convexity identity

\[
\|(1 - t)u + tv\|_-^2 + t(1 - t)\|v - u\|_-^2 = (1 - t)\|u\|_-^2 + t\|v\|_-^2
\]

holds for all elements \( u \) and \( v \) of the space when \( 0 < t < 1 \). Use is made of the convexity identity

\[
\langle(1 - t)a + tb, (1 - t)u + tv\rangle + t(1 - t)\langle b - a, v - u\rangle = (1 - t)\langle a, u\rangle + t\langle b, v\rangle
\]

for elements \( a, b, u, \) and \( v \) of the space when \( 0 < t < 1 \). Since the inequality

\[
|(1 - t)\langle a, u\rangle + t\langle b, v\rangle| \\
\leq \|(1 - t)a + t b\|_+\|(1 - t)u + tv\|_- + t(1 - t)\|b - a\|_+\|v - u\|_-
\]

holds by the definition of the minus norm, the inequality

\[
|(1 - t)\langle a, u\rangle + t\langle b, v\rangle|^2 \leq \left[\|(1 - t)a + tb\|_+^2 + t(1 - t)\|b - a\|_+^2\right] \\
\times\left[\|(1 - t)u + tv\|_-^2 + t(1 - t)\|v - u\|_-^2\right]
\]

is satisfied. The inequality

\[
|(1 - t)\langle a, u\rangle + t\langle b, v\rangle|^2 \leq \left[(1 - t)\|a\|_+^2 + t\|b\|_+^2\right] \\
\times\left[\|(1 - t)u + tv\|_-^2 + t(1 - t)\|v - u\|_-^2\right]
\]

holds by the convexity identity for the plus norm. The inequality is applied for all elements \( a \) and \( b \) of the space such that the inequalities

\[
\|a\|_+ \leq \|u\|_- 
\]
and

\[ \|b\|_+ \leq \|v\|_- \]

are satisfied. The inequality

\[(1 - t)\|u\|_+^2 + t\|v\|_+^2 \leq \|(1 - t)u + tv\|_+^2 + t(1 - t)\|v - u\|_-^2 \]

follows by the definition of the minus norm. Equality holds since the reverse inequality is a consequence of the identities

\[ (1 - t)[(1 - t)u + tv] + t[(1 - t)u - (1 - t)v] = (1 - t)u \]

and

\[ [(1 - t)u + tv] - [(1 - t)u - (1 - t)v] = v. \]

It has been verified that the minus norm satisfies the hypotheses of the theorem. The dual norm to the minus norm is the plus norm. Another norm which satisfies the hypotheses of the theorem is defined by

\[ \|c\|_t^2 = (1 - t)\|c\|_+^2 + t\|c\|_-^2 \]

when \(0 < t < 1\). Since the inequalities

\[ |\langle a, b \rangle| \leq \|a\|_+ \|b\|_- \]

and

\[ |\langle a, b \rangle| \leq \|a\|_- \|b\|_+ \]

hold for all elements \(a\) and \(b\) of the space, the inequality

\[ |\langle a, b \rangle| \leq (1 - t)\|a\|_+ \|b\|_- + t\|a\|_- \|b\|_+ \]

holds when \(0 < t < 1\). The inequality

\[ |\langle a, b \rangle| \leq \|a\|_t \|b\|_{1-t} \]

follows for all elements \(a\) and \(b\) of the space when \(0 < t < 1\). The inequality implies that the dual norm of the \(t\) norm is dominated by the \(1 - t\) norm. A norm which dominates its dual norm is obtained when \(t = \frac{1}{2}\).

Consider the norms which satisfy the hypotheses of the theorem and which dominate their dual norms. Since a nonempty totally ordered set of such norms has a greatest lower bound, which is again such a norm, a minimal such norm exists by the Zorn lemma. If a minimal norm is chosen as the plus norm, it is equal to the \(t\)-norm obtained when \(t = \frac{1}{2}\). It follows that a minimal norm is equal to its dual norm.

If a norm satisfies the hypotheses of the theorem and is equal to its dual norm, a related scalar product is introduced on the space which may be different from the given
scalar product. Since the given scalar product assumes a subsidiary role in the subsequent argument, it is distinguished by a prime. A new scalar product is defined by the identity

$$4\langle a, b \rangle = \|a + b\|^2 - \|a - b\|^2 + i\|a + ib\|^2 - i\|a - ib\|^2.$$ 

The symmetry of a scalar product is immediate. Linearity will be verified.

The identity

$$\langle wa, wb \rangle = w^* w \langle a, b \rangle$$

holds for all elements $a$ and $b$ of the space if $w$ is a complex number. The identity

$$\langle ia, b \rangle = i \langle a, b \rangle$$

holds for all elements $a$ and $b$ of the space. The identity

$$\langle ta, b \rangle = t \langle a, b \rangle$$

will be verified for all elements $a$ and $b$ of the space when $t$ is a positive number. It is sufficient to verify the identity

$$\|ta + b\|^2 - \|ta - b\|^2 = t\|a + b\|^2 - t\|a - b\|^2$$

since a similar identity follows with $b$ replaced by $ib$. The identity holds since

$$\|ta + b\|^2 + t\|a - b\|^2 = t(1 + t)\|a\|^2 + (1 + t)\|b\|^2$$

and

$$\|ta - b\|^2 + t\|a + b\|^2 = t(1 + t)\|a\|^2 + (1 + t)\|b\|^2$$

by the convexity identity.

If $a, b,$ and $c$ are elements of the space and if $0 < t < 1$, the identity

$$4\langle (1 - t)a + tb, c \rangle = \|(1 + t)(a + c) + t(b + c)\|^2$$

$$-\|(1 - t)(a - c) + t(b - c)\|^2 + i\|(1 - t)(a + ic) + t(b + ic)\|^2$$

$$-i\|(1 - t)(a - ic) + t(b - ic)\|^2$$

is satisfied with the right side equal to

$$(1 - t)\|a + c\|^2 + t\|b + c\|^2 - (1 - t)\|a - c\|^2 - t\|b - c\|^2$$

$$+i(1 - t)\|a + ic\|^2 + it\|b + ic\|^2 - i(1 - t)\|a - ic\|^2 - it\|b - ic\|^2$$

$$= 4(1 - t)\langle a, c \rangle + 4t\langle b, c \rangle.$$ 

The identity

$$\langle (1 - t)a + tb, c \rangle = (1 - t)\langle a, c \rangle + t\langle b, c \rangle$$

follows.
Linearity of a scalar product is now easily verified. Scalar self-products are nonnegative since the identity

$$\langle c, c \rangle = \|c\|^2$$

holds a for every element $c$ of the space. A Hilbert space is obtained whose norm is the minimal norm. Since the inequality

$$|\langle a, b \rangle'| \leq \|a\| \|b\|$$

holds for all elements $a$ and $b$ of the space, a contractive transformation $J$ of the Hilbert space into itself exists such that the identity

$$\langle a, b \rangle' = \langle Ja, b \rangle$$

holds for all elements $a$ and $b$ of the space. The symmetry of the given scalar product implies that the transformation $J$ is self-adjoint. Since the Hilbert space norm is self-dual with respect to the given scalar product, the transformation $J$ is also isometric with respect to the Hilbert space scalar product. The space is the orthogonal sum of the space of eigenvectors of $J$ for the eigenvalue one and the space of eigenvectors of $J$ for the eigenvalue minus one. These spaces are also orthogonal with respect to the given scalar product. They are the required Hilbert space and anti-space of a Hilbert space for the orthogonal decomposition of the vector space with scalar product to form a Krein space.

This completes the proof of the theorem.

The orthogonal decomposition of a Krein space is not unique since equivalent norms can be used. The dimension of the anti-space of a Hilbert space in the decomposition is however an invariant called the Pontryagin index of the Krein space. Krein spaces are a natural context for a complementation theory which was discovered in Hilbert spaces [3].

A generalization of the concept of orthogonal complement applies when a Krein space $\mathcal{P}$ is contained continuously and contractively in a Krein space $\mathcal{H}$. The contractive property of the inclusion means that the inequality

$$\langle a, a \rangle_{\mathcal{H}} \leq \langle a, a \rangle_{\mathcal{P}}$$

holds for every element $a$ of $\mathcal{P}$. Continuity of the inclusion means that an adjoint transformation of $\mathcal{H}$ into $\mathcal{P}$ exists. A self-adjoint transformation $\mathcal{P}$ of $\mathcal{H}$ into $\mathcal{H}$ is obtained on composing the inclusion with the adjoint. The inequality

$$\langle Pc, Pc \rangle_{\mathcal{H}} \leq \langle Pc, Pc \rangle_{\mathcal{P}}$$

for elements $c$ of $\mathcal{H}$ implies the inequality

$$\langle P^2 c, c \rangle_{\mathcal{H}} \leq \langle Pc, c \rangle_{\mathcal{H}}$$

for elements $c$ of $\mathcal{H}$, which is restated as an inequality

$$P^2 \leq P$$

for self-adjoint transformations in $\mathcal{H}$.

The properties of adjoint transformations are used in the construction of a complementary space $\mathcal{Q}$ to $\mathcal{P}$ in $\mathcal{H}$. 
Theorem 2. If a Krein space $\mathcal{P}$ is contained continuously and contractively in a Krein space $\mathcal{H}$, then a unique Krein space $\mathcal{Q}$ exists, which is contained continuously and contractively in $\mathcal{H}$, such that the inequality

$$\langle c, c \rangle_{\mathcal{H}} \leq \langle a, a \rangle_{\mathcal{P}} + \langle b, b \rangle_{\mathcal{Q}}$$

holds whenever $c = a + b$ with $a$ in $\mathcal{P}$ and $b$ in $\mathcal{Q}$ and such that every element $c$ of $\mathcal{H}$ admits some such decomposition for which equality holds.

Proof. Define $\mathcal{Q}$ to be the set of elements $b$ of $\mathcal{H}$ such that the least upper bound

$$\langle b, b \rangle_{\mathcal{Q}} = \sup \{ \langle a + b, a + b \rangle_{\mathcal{H}} - \langle a, a \rangle_{\mathcal{P}} \}$$

taken over all elements $a$ of $\mathcal{P}$ is finite. It will be shown that $\mathcal{Q}$ is a vector space with scalar product having the desired properties. Since the origin belongs to $\mathcal{P}$, the inequality

$$\langle b, b \rangle_{\mathcal{H}} \leq \langle b, b \rangle_{\mathcal{Q}}$$

holds for every element $b$ of $\mathcal{Q}$. Since the inclusion of $\mathcal{P}$ in $\mathcal{H}$ is contractive, the origin belongs to $\mathcal{Q}$ and has self-product zero. If $b$ belongs to $\mathcal{Q}$ and if $w$ is a complex number, then $wb$ is an element of $\mathcal{Q}$ which satisfies the identity

$$\langle wb, wb \rangle_{\mathcal{Q}} = w \langle b, b \rangle_{\mathcal{Q}}.$$
is satisfied.

This completes the verification that \( \mathcal{Q} \) is a vector space. It will be shown that a scalar product is defined on the space by the identity

\[
4\langle a, b \rangle_{\mathcal{Q}} = \langle a + b, a + b \rangle_{\mathcal{Q}} - \langle a - b, a - b \rangle_{\mathcal{Q}} + i\langle a + ib, a + ib \rangle_{\mathcal{Q}} - i\langle a - ib, a - ib \rangle_{\mathcal{Q}}.
\]

Linearity and symmetry of a scalar product are verified as in the characterization of Krein spaces. The nondegeneracy of a scalar product remains to be verified.

Since the inclusion of \( \mathcal{P} \) in \( \mathcal{H} \) is continuous, a self-adjoint transformation \( P \) of \( \mathcal{H} \) into itself exists which coincides with the adjoint of the inclusion of \( \mathcal{P} \) in \( \mathcal{H} \). If \( c \) is an element of \( \mathcal{H} \) and if \( a \) is an element of \( \mathcal{P} \), the inequality

\[
\langle a - Pc, a - Pc \rangle_{\mathcal{H}} \leq \langle a - Pc, a - Pc \rangle_{\mathcal{P}}
\]

implies the inequality

\[
\langle (1 - P)c, (1 - P)c \rangle_{\mathcal{Q}} \leq \langle c, c \rangle_{\mathcal{H}} - \langle Pc, Pc \rangle_{\mathcal{P}}.
\]

Equality holds since the reverse inequality follows from the definition of the self-product in \( \mathcal{Q} \). If \( b \) is an element of \( \mathcal{Q} \) and if \( c \) is an element of \( \mathcal{H} \), the inequality

\[
\langle b - c, b - c \rangle_{\mathcal{H}} \leq \langle Pc, Pc \rangle_{\mathcal{P}} + \langle b - (1 - P)c, b - (1 - P)c \rangle_{\mathcal{Q}}
\]

can be written

\[
\langle b, b \rangle_{\mathcal{H}} - \langle b, c \rangle_{\mathcal{H}} - \langle c, b \rangle_{\mathcal{H}} \leq \langle b, b \rangle_{\mathcal{Q}} - \langle b, (1 - P)c \rangle_{\mathcal{Q}} - \langle (1 - P)c, b \rangle_{\mathcal{Q}}.
\]

Since \( b \) can be replaced by \( wb \) for every complex number \( w \), the identity

\[
\langle b, c \rangle_{\mathcal{H}} = \langle b, (1 - P)c \rangle_{\mathcal{Q}}
\]

is satisfied. The nondegeneracy of a scalar product follows in the space \( \mathcal{Q} \). The space \( \mathcal{Q} \) is contained continuously in the space \( \mathcal{H} \) since \( 1 - P \) coincides with the adjoint of the inclusion of \( \mathcal{Q} \) in the space \( \mathcal{H} \).

The intersection of \( \mathcal{P} \) and \( \mathcal{Q} \) is considered as a vector space \( \mathcal{P} \cap \mathcal{Q} \) with scalar product

\[
\langle a, b \rangle_{\mathcal{P} \cap \mathcal{Q}} = \langle a, b \rangle_{\mathcal{P}} + \langle a, b \rangle_{\mathcal{Q}}.
\]

Linearity and symmetry of a scalar product are immediate, but nondegeneracy requires verification. If \( c \) is an element of \( \mathcal{H} \),

\[
P(1 - P)c = (1 - P)Pc
\]

is an element of \( \mathcal{P} \cap \mathcal{Q} \) which satisfies the identity

\[
\langle a, P(1 - P)c \rangle_{\mathcal{P} \cap \mathcal{Q}} = \langle a, c \rangle_{\mathcal{H}}
\]
for every element $a$ of $\mathcal{P} \wedge \mathcal{Q}$. Nondegeneracy of a scalar product in $\mathcal{P} \wedge \mathcal{Q}$ follows from nondegeneracy of the scalar product in $\mathcal{H}$. The space $\mathcal{P} \wedge \mathcal{Q}$ is contained continuously in the space $\mathcal{H}$. The self-adjoint transformation $P(1 - P)$ in $\mathcal{H}$ coincides with the adjoint of the inclusion of $\mathcal{P} \wedge \mathcal{Q}$ in $\mathcal{H}$. The inequality

$$0 \leq \langle c, c \rangle_{\mathcal{P} \wedge \mathcal{Q}}$$

holds for every element $c$ of $\mathcal{P} \wedge \mathcal{Q}$ since the identity

$$0 = c - c$$

with $c$ in $\mathcal{P}$ and $-c$ in $\mathcal{Q}$ implies the inequality

$$0 \leq \langle c, c \rangle_{\mathcal{P}} + \langle c, c \rangle_{\mathcal{Q}}.$$ 

It will be shown that the space $\mathcal{P} \wedge \mathcal{Q}$ is a Hilbert space. The metric topology of the space is the disk topology resulting from duality of the space with itself. Since the inclusion of $\mathcal{P} \wedge \mathcal{Q}$ in $\mathcal{P}$ is continuous from the weak topology induced by $\mathcal{P} \wedge \mathcal{Q}$ into the weak topology induced by $\mathcal{P}$, it is continuous from the disk topology induced by $\mathcal{P} \wedge \mathcal{Q}$ into the disk topology induced by $\mathcal{P}$. Since $\mathcal{P}$ is a Krein space, it is complete in its disk topology. A Cauchy sequence of elements $c_n$ of $\mathcal{P} \wedge \mathcal{Q}$ is then a convergent sequence of elements of $\mathcal{P}$. The limit is an element $c$ of $\mathcal{P}$ such that the identity

$$\langle c, a \rangle_{\mathcal{P}} = \lim \langle c_n, a \rangle_{\mathcal{P}}$$

holds for every element $a$ of $\mathcal{P}$ and such that the identity

$$\langle c, c \rangle_{\mathcal{P}} = \lim \langle c_n, c_n \rangle_{\mathcal{P}}$$

is satisfied. Since the inclusion of $\mathcal{P}$ in $\mathcal{H}$ is continuous from the disk topology of $\mathcal{P}$ into the disk topology of $\mathcal{H}$, the identity

$$\langle c, a \rangle_{\mathcal{H}} = \lim \langle c_n, a \rangle_{\mathcal{H}}$$

holds for every element $a$ of $\mathcal{H}$ and the identity

$$\langle c, c \rangle_{\mathcal{H}} = \lim \langle c_n, c_n \rangle_{\mathcal{H}}$$

is satisfied.

If $b$ is an element of $\mathcal{Q}$, the limits

$$\lim \langle c_n, b \rangle_{\mathcal{Q}}$$

and

$$\lim \langle c_n, c_n \rangle_{\mathcal{Q}}$$
exist since the inclusion of $P \land Q$ in $Q$ is continuous from the disk topology of $P \land Q$ into the disk topology of $Q$. The sequence of elements $c_n$ of $Q$ is Cauchy in the disk topology of $Q$. If $a$ is an element of $P$, the identity

$$\langle a + c, a + c \rangle_H = \lim \langle a + c_n, a + c_n \rangle_H$$

is satisfied. Since the inequality

$$\langle a + c_n, a + c_n \rangle_H - \langle a, a \rangle_P \leq \langle c_n, c_n \rangle_Q$$

holds for every index $n$, the inequality

$$\langle a + c, a + c \rangle_H - \langle a, a \rangle_P \leq \lim \langle c_n, c_n \rangle_Q$$

is satisfied. It follows that $c$ belongs to $Q$ and that

$$\langle c, c \rangle_Q \leq \lim \langle c_n, c_n \rangle_Q.$$ 

Since the inequality

$$\langle c - c_m, c - c_m \rangle_Q \leq \lim \langle c_n - c_m, c_n - c_m \rangle_Q$$

holds for every index $m$ and since the elements $c_n$ of $Q$ form a Cauchy sequence in the disk topology of $Q$, the limit of the elements $c_n$ of $Q$ is equal to $c$. This completes the proof that $P \land Q$ is a Hilbert space.

The Cartesian product of $P$ and $Q$ is isomorphic to the Cartesian product of $H$ and $P \land Q$. If $a$ is an element of $P$ and if $b$ is an element of $Q$, a unique element $c$ of $P \land Q$ exists such that the identity

$$\langle a - c, a - c \rangle_P + \langle b + c, b + c \rangle_Q = \langle a + b, a + b \rangle_H + \langle c, c \rangle_{P \land Q}$$

is satisfied. Every element of the Cartesian product of $H$ and $P \land Q$ is a pair $(a + b, c)$ for elements $a$ of $P$ and $b$ of $Q$ for such an element $c$ of $P \land Q$. Since $H$ is a Krein space and since $P \land Q$ is a Hilbert space, the Cartesian product of $P$ and $Q$ is a Krein space. Since $P$ is a Krein space, it follows that $Q$ is a Krein space.

The existence of a Krein space $Q$ with the desired properties has now been verified. Uniqueness is proved by showing that a Krein space $Q'$ with these properties is isometrically equal to the space $Q$ constructed. Such a space $Q'$ is contained contractively in the space $Q$. The self-adjoint transformation $1 - P$ in $H$ coincides with the adjoint of the inclusion of $Q'$ in $H$. The space $P \land Q'$ is a Hilbert space which is contained contractively in the Hilbert space $P \land Q$. Since the inclusion is isometric on the range of $P(1 - P)$, which is dense in both spaces, the space $P \land Q'$ is isometrically equal to the space $P \land Q$. Since the Cartesian product of $P$ and $Q'$ is isomorphic to the Cartesian product of $P$ and $Q$, the spaces $Q$ and $Q'$ are isometrically equal.

This completes the proof of the theorem.
The space \(Q\) is called the complementary space to \(P\) in \(\mathcal{H}\). The space \(P\) is recovered as the complementary space to the space \(Q\) in \(\mathcal{H}\). The decomposition of an element \(c\) of \(\mathcal{H}\) as \(c = a + b\) with \(a\) an element of \(P\) and \(b\) an element of \(Q\) such that equality hold in the inequality

\[
\langle c, c \rangle_{\mathcal{H}} \leq \langle a, a \rangle_P + \langle b, b \rangle_Q
\]

is unique. The minimal decomposition results when \(a\) is obtained from \(c\) under the adjoint of the inclusion of \(P\) in \(\mathcal{H}\) and \(b\) is obtained from \(c\) under the adjoint of the inclusion of \(Q\) in \(\mathcal{H}\).

A construction is made of complementary subspaces whose inclusion in the full space have adjoints coinciding with given self-adjoint transformations.

**Theorem 3.** If a self-adjoint transformation \(P\) of a Krein space into itself satisfies the inequality

\[
P^2 \leq P,
\]

then unique Krein spaces \(P\) and \(Q\) exist, which are contained continuously and contractively in \(\mathcal{H}\) and which are complementary spaces in \(\mathcal{H}\), such that \(P\) coincides with the adjoint of the inclusion of \(P\) in \(\mathcal{H}\) and \(1 - P\) coincides with the adjoint of the inclusion of \(Q\) in \(\mathcal{H}\).

**Proof of Theorem 3.** The proof repeats the construction of a complementary space under a weaker hypothesis. The range of \(P\) is considered as a vector space \(P'\) with scalar product determined by the identity

\[
\langle Pc, Pc \rangle_{P'} = \langle Pc, c \rangle_{\mathcal{H}},
\]

for every element \(c\) of \(\mathcal{H}\). The space \(P'\) is contained continuously and contractively in the space \(\mathcal{H}\). The transformation \(P\) coincides with the adjoint of the inclusion of \(P'\) in \(\mathcal{H}\). A Krein space \(Q\), which is contained continuously and contractively in \(\mathcal{H}\), is defined as the set of elements \(b\) of \(\mathcal{H}\) such that the least upper bound

\[
\langle b, b \rangle_Q = \sup \{\langle a + b, a + b \rangle_{\mathcal{H}} - \langle a, a \rangle_{P'}\}
\]

taken over all elements \(a\) of \(P'\) is finite. The adjoint of the inclusion of \(Q\) in \(\mathcal{H}\) coincides with \(1 - P\). The complementary space to \(Q\) in \(\mathcal{H}\) is a Krein space \(P\) which contains the space \(P'\) isometrically and which is contained continuously and contractively in \(\mathcal{H}\). The adjoint of the inclusion of \(P\) in \(\mathcal{H}\) coincides with \(1 - P\).

This completes the proof of the theorem.

A factorization of continuous and contractive transformations in Krein spaces is an application of complementation theory.

**Theorem 4.** The kernel of a continuous and contractive transformation \(T\) of a Krein space \(P\) into a Krein space \(Q\) is a Hilbert space which is contained continuously and isometrically in \(P\) and whose orthogonal complement in \(P\) is mapped isometrically onto a Krein space which is contained continuously and contractively in \(Q\).
Proof of Theorem 4. Since the transformation $T$ of $\mathcal{P}$ into $\mathcal{Q}$ is continuous and contractive, the self-adjoint transformation $P = TT^*$ in $\mathcal{Q}$ satisfies the inequality $P^2 \leq P$. A unique Krein space $\mathcal{M}$, which is contained continuously and contractively in $\mathcal{Q}$, exists such that $\mathcal{P}$ coincides with the adjoint of the inclusion of $\mathcal{M}$ in $\mathcal{Q}$. It will be shown that $T$ maps $\mathcal{P}$ contractively into $\mathcal{M}$.

If $a$ is an element of $\mathcal{P}$ and if $b$ is an element of $\mathcal{Q}$, then

$$\langle Ta + (1 - P)b, Ta + (1 - P)b \rangle_{\mathcal{Q}}$$

$$= \langle T(a - T^*b), T(a - T^*b)(\mathcal{Q}+), b, b \rangle_{\mathcal{Q}} + \langle b, T(a - T^*b) \rangle_{\mathcal{Q}} + \langle T(a - T^*b), b \rangle_{\mathcal{Q}}$$

is less than or equal to

$$\langle a - T^*b, a - T^*b \rangle_{\mathcal{P}} + \langle b, b \rangle_{\mathcal{Q}} + \langle T^*b, a - T^*b \rangle_{\mathcal{P}} + \langle a - T^*b, T^*b \rangle_{\mathcal{P}}$$

$$= \langle a, a \rangle_{\mathcal{P}} + \langle (1 - TT^*)b, b \rangle_{\mathcal{Q}}.$$ 

Since $b$ is an arbitrary element of $\mathcal{Q}$, $Ta$ is an element of $\mathcal{M}$ which satisfies the inequality

$$\langle Ta, Ta \rangle_{\mathcal{M}} \leq \langle a, a \rangle_{\mathcal{P}}.$$ 

Equality holds when $a = T^*b$ for an element $b$ of $\mathcal{Q}$ since

$$\langle TT^*b, TT^*b \rangle_{\mathcal{M}} = \langle TT^*b, b \rangle_{\mathcal{Q}} = \langle T^*b, T^*b \rangle_{\mathcal{P}}.$$ 

Since the transformation of $\mathcal{P}$ into $\mathcal{M}$ is continuous by the closed graph theorem, the adjoint transformation is an isometry. The range of the adjoint transformation is a Krein space which is contained continuously and isometrically in $\mathcal{P}$ and whose orthogonal complement is the kernel of $T$. Since $T$ is contractive, the kernel of $T$ is a Hilbert space.

This completes the proof of the theorem.

A continuous transformation of a Krein space $\mathcal{P}$ into a Krein space $\mathcal{Q}$ is said to be a partial isometry if its kernel is a Krein space which is contained continuously and isometrically in $\mathcal{P}$ and whose orthogonal complement is mapped isometrically into $\mathcal{Q}$. A partially isometric transformation of a Krein space into a Krein space is contractive if, and only if, its kernel is a Hilbert space. Complementation is preserved under contractive partially isometric transformations of a Krein space onto a Krein space.

**Theorem 5.** If a contractive partially isometric transformation $T$ maps a Krein space $\mathcal{H}$ onto a Krein space $\mathcal{H}'$ and if Krein spaces $\mathcal{P}$ and $\mathcal{Q}$ are contained continuously and contractively as complementary subspaces of $\mathcal{H}$, then Krein spaces $\mathcal{P}'$ and $\mathcal{Q}'$, which are contained continuously and contractively as complementary subspaces of $\mathcal{H}'$, exist such that $T$ acts as a contractive partially isometric transformation of $\mathcal{P}$ onto $\mathcal{P}'$ and of $\mathcal{Q}$ onto $\mathcal{Q}'$.

**Proof of Theorem 5.** Since the Krein spaces $\mathcal{P}$ and $\mathcal{Q}$ are contained continuously and contractively in $\mathcal{H}$ and since $T$ is a continuous and contractive transformation of $\mathcal{H}$ into
$\mathcal{H}'$, $T$ acts as a continuous and contractive transformation of $\mathcal{P}$ into $\mathcal{H}'$ and of $\mathcal{Q}$ into $\mathcal{H}'$. Krein spaces $\mathcal{P}'$ and $\mathcal{Q}'$, which are contained continuously and contractively in $\mathcal{H}'$, exist such that $T$ acts as a contractive partially isometric transformation of $\mathcal{P}$ onto $\mathcal{P}'$ and of $\mathcal{Q}$ onto $\mathcal{Q}'$. It will be shown that $\mathcal{P}'$ and $\mathcal{Q}'$ are complementary subspaces of $\mathcal{H}'$.

An element $a$ of $\mathcal{P}'$ is of the form $Ta$ for an element $a$ of $\mathcal{P}$ such that

$$\langle Ta, Ta \rangle_{\mathcal{P}'} = \langle a, a \rangle_{\mathcal{P}}.$$

An element $b$ of $\mathcal{Q}'$ is of the form $Tb$ for an element $b$ of $\mathcal{Q}$ such that

$$\langle Tb, Tb \rangle_{\mathcal{Q}'} = \langle b, b \rangle_{\mathcal{Q}}.$$

The element $c = a + b$ of $\mathcal{H}$ satisfies the inequalities

$$\langle c, c \rangle_{\mathcal{H}} \leq \langle a, a \rangle_{\mathcal{P}} + \langle b, b \rangle_{\mathcal{Q}}$$

and

$$\langle Tc, Tc \rangle_{\mathcal{H}'} \leq \langle c, c \rangle_{\mathcal{H}}.$$

The element $Tc = Ta + Tb$ of $\mathcal{H}'$ satisfies the inequality

$$\langle Tc, Tc \rangle_{\mathcal{H}'} \leq \langle Ta, Ta \rangle_{\mathcal{P}'} + \langle Tb, Tb \rangle_{\mathcal{Q}'}.$$

An element of $\mathcal{H}'$ is of the form $Tc$ for an element $c$ of $\mathcal{H}$ such that

$$\langle Tc, Tc \rangle_{\mathcal{H}'} = \langle c, c \rangle_{\mathcal{H}}.$$

An element $a$ of $\mathcal{P}$ and an element $b$ of $\mathcal{Q}$ exist such that $c = a + b$ and

$$\langle c, c \rangle_{\mathcal{H}} = \langle a, a \rangle_{\mathcal{P}} + \langle b, b \rangle_{\mathcal{Q}}.$$

Since the element $Ta$ of $\mathcal{P}'$ satisfies the inequality

$$\langle Ta, Tb \rangle_{\mathcal{P}'} \leq \langle a, a \rangle_{\mathcal{P}}$$

and since the element $Tb$ of $\mathcal{Q}'$ satisfies the inequality

$$\langle Tb, Tb \rangle_{\mathcal{Q}'} \leq \langle b, b \rangle_{\mathcal{Q}},$$

the element $Tc$ of $\mathcal{H}$ satisfies the inequality

$$\langle Tc, Tc \rangle_{\mathcal{H}'} \geq \langle Ta, Ta \rangle_{\mathcal{P}'} + \langle Tb, Tb \rangle_{\mathcal{Q}'}.$$

Equality holds since the reverse inequality is satisfied.

This completes the proof of the theorem.
A canonical coisometric linear system whose state space is a Hilbert space is constructed when multiplication by \( W(z) \) is a contractive transformation in \( C(z) \). The range of multiplication by \( W(z) \) in \( C(z) \) is a Hilbert space which is contained contractively in \( C(z) \) when considered with the unique scalar product such that multiplication by \( W(z) \) acts as a partially isometric transformation of \( C(z) \) onto the range. The complementary space in \( C(z) \) to the range is the state space \( H(W) \) of a canonical coisometric linear system with transfer function \( W(z) \). Every Hilbert space which is the state space of a canonical coisometric linear system is so obtained.

A Herglotz space is a Hilbert space, whose elements are power series, such that the difference-quotient transformation is a continuous transformation of the space into itself which has an isometric adjoint and such that a continuous transformation of the space into the space of complex numbers is defined by taking \( f(z) \) into \( f(0) \). A continuous transformation of the space into the space of complex numbers is then defined by taking a power series into its coefficient of \( z^n \) for every nonnegative integer \( n \). A Herglotz function for the space is a power series \( z^n z^n \) such that the adjoint of the continuous transformation of the space into the complex numbers takes a complex number \( c \) into

\[
\frac{1}{2} \left[ z^n \phi(z) + \phi(c) z^n + \cdots + \phi(0) \right] c.
\]

A Herglotz function for the space is determined within an added imaginary constant by the adjoint computation when \( n \) is zero. The form of the adjoint for positive integers \( n \) is verified inductively using the isometric property of the adjoint of the difference-quotient transformation. The adjoint transformation takes \( f(z) \) into \( zf(z) + c \) for a vector \( c \) which depends continuously on \( f(z) \) and which is computed inductively in the present application. A Herglotz space is uniquely determined by its Herglotz function.

The Herglotz function of a Herglotz space is a power series \( \phi(z) \) which represents a function whose values in the unit disk have nonnegative real part. The Poisson representation of \( \phi(z) \) is an integral

\[
\phi(z) = \sum \phi_n z^n
\]

such that the adjoint of the continuous transformation of the space into the complex numbers takes a complex number \( c \) into

\[
\frac{1}{2} \left[ z^n \phi(z) + \phi(c) z^n + \cdots + \phi(0) \right] c.
\]

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A Herglotz function for the space is determined within an added imaginary constant by the adjoint computation when \( n \) is zero. The form of the adjoint for positive integers \( n \) is verified inductively using the isometric property of the adjoint of the difference-quotient transformation. The adjoint transformation takes \( f(z) \) into \( zf(z) + c \) for a vector \( c \) which depends continuously on \( f(z) \) and which is computed inductively in the present application. A Herglotz space is uniquely determined by its Herglotz function.
whenever it takes the function $f(\theta)$ of $\theta$ into the power series $g(z)$. The identity

$$2\pi \langle g(z), g(z) \rangle_{L(\phi)} = \int f(\theta)^{-1} d\mu(\theta) f(\theta)$$

holds when the function $f(\theta)$ of $\theta$ in the real numbers modulo $2\pi$ is orthogonal to the kernel of the transformation.

The extension space $\text{ext } L(\phi)$ of the Herglotz space $L(\phi)$ is a Hilbert space of Laurent series, which is invariant under division by $z$, such that the canonical projection onto the space $L(\phi)$ is a partial isometry. The canonical projection takes a Laurent series into the power series which has the same coefficient of $z^n$ for every nonnegative integer $n$. Uniqueness of the extension space results from the hypothesis that an element $f(z)$ in the real numbers modulo 2 is orthogonal to the kernel of the transformation.

The spectral subspace of contractivity is constructed for a closed relation $T$ whose domain is contained in a Hilbert space $P$ and whose range is contained in a Hilbert space $Q$. The relation $T$ is then the adjoint of the adjoint relation $T^*$ which has its domain contained in the Hilbert space $Q$ and its range contained in the Hilbert space $P$. A self-adjoint relation $H$ in the Cartesian product Hilbert space $P \times Q$ is defined by taking $(a, b)$ into $(T^* b, Ta)$ when $a$ is in the domain of $T$ and $b$ is in the domain of $T^*$. The spectral subspace of contractivity for $H$ is a closed subspace of the Cartesian product such that $H$ acts as a contractive transformation of the subspace into itself. The orthogonal complement is a closed subspace such that the inverse of $H$ acts as a contractive transformation of the subspace into itself. Eigenvectors of $H$ for eigenvalues of absolute value one belong to the spectral subspace of contractivity for $H$. The square of $H$ is a self-adjoint relation in the Cartesian product space which has the same spectral subspace of contractivity. Since the transformation which takes $(a, b)$ into $(a, -b)$ commutes with the square of $H$, the spectral subspace of contractivity for $H$ is the Cartesian product of a closed subspace of $P$ and a closed subspace of $Q$. The spectral subspace of contractivity for $T$ is the closed subspace of $P$. The spectral subspace of contractivity for $T^*$ is the closed subspace of $Q$. The relation $T$ acts as a contractive transformation of the spectral subspace of contractivity for $T$ into the spectral subspace of contractivity for $T^*$. The relation $T^*$ acts as a contractive transformation of the spectral subspace of contractivity for $T^*$ into the spectral subspace of
contractivity for $T$. The inverse of $T$ acts as a contractive transformation of the orthogonal complement of the spectral subspace of contractivity for $T^*$ into the orthogonal complement of the spectral subspace of contractivity for $T$. The inverse of $T^*$ acts as a contractive transformation of the orthogonal complement of the spectral subspace of contractivity for $T$ into the orthogonal complement of the spectral subspace of contractivity for $T^*$. If $a$ is an element of $\mathcal{P}$ and if $b$ is an element of $\mathcal{Q}$ such that the identities

$$Ta = b$$

and

$$T^*b = a$$

are satisfied, then $a$ belongs to the spectral subspace of contractivity for $T$ and $b$ belongs to the spectral subspace of contractivity for $T^*$.

A Herglotz space is associated with the transfer function $W(z)$ of a canonical coisometric linear system whose state space is a Hilbert space. Since multiplication by $W(z)$ is an everywhere defined and contractive transformation in $\mathcal{C}(z)$, the adjoint of multiplication by $W(z)$ is an everywhere defined and contractive transformation in $\mathcal{C}(z)$. The range of the adjoint is a Hilbert space which is contained contractively in $\mathcal{C}(z)$ when it is considered with the scalar product such that the adjoint acts as a partially isometric transformation of $\mathcal{C}(z)$ onto the range. The space is a Herglotz space. The space $\mathcal{C}(z)$ is a Herglotz space $\mathcal{L}(1)$ whose Herglotz function is identically one. The complementary space to the range Herglotz space is a Herglotz space whose Herglotz function $\phi(z)$ is determined within an added constant, which is a skew-conjugate operator, by the identity

$$\phi(z) + \phi^*(z^{-1}) = 2 - 2W^*(z^{-1})W(z).$$

The space $\mathcal{L}(\phi)$ is the set of elements $f(z)$ of $\mathcal{C}(z)$ such that $W(z)f(z)$ belongs to the space $\mathcal{H}(W)$. The scalar product in the space $\mathcal{L}(\phi)$ is determined by the identity

$$\langle f(z), f(z) \rangle_{\mathcal{L}(\phi)} = \langle f(z), f(z) \rangle_{\mathcal{C}(z)} + \langle W(z)f(z), W(z)f(z) \rangle_{\mathcal{H}(W)}.$$

The adjoint of multiplication by $W(z)$ as a transformation in $\mathcal{C}(z)$ acts as a partially isometric transformation of $\mathcal{C}(z)$ onto the complementary space $\mathcal{L}(1 - \phi)$ to $\mathcal{L}(\phi)$ in $\mathcal{C}(z)$. Since the polynomial elements of $\mathcal{C}(z)$ are dense in $\mathcal{C}(z)$, the polynomial elements of the space $\mathcal{L}(1 - \phi)$ are dense in the space $\mathcal{C}(z)$.

If a Herglotz space $\mathcal{L}$ is contained contractively in $\mathcal{C}(z)$ and if the polynomial elements of the space are dense in the space, then a partially isometric transformation of $\mathcal{C}(z)$ onto $\mathcal{L}$ exists which commutes with the difference-quotient transformation and whose kernel is invariant under multiplication by $z$. The resulting contractive transformation of $\mathcal{C}(z)$ into itself coincides with the adjoint of multiplication by $V(z)$ for a power series $V(z)$ with complex coefficients and that multiplication by $V(z)$ is an everywhere defined and contractive transformation in $\mathcal{C}(z)$ and such that the orthogonal complement of the range of multiplication by $V(z)$ in $\mathcal{C}(z)$ is invariant under multiplication by $z$.

The construction of $V(z)$ is supplied when $W(z)$ is a power series such that multiplication by $W(z)$ is an everywhere defined and contractive transformation in $\mathcal{C}(z)$. The
assofated Herglotz space $\mathcal{L}(\phi)$ contains the elements $f(z)$ of $\mathcal{C}(z)$ such that $W(z)f(z)$ belongs to the space $\mathcal{H}(W)$. The identity

$$\|f(z)\|^2_{\mathcal{L}(\phi)} = \|f(z)\|^2_{\mathcal{C}(z)} + \|W(z)f(z)\|^2_{\mathcal{H}(W)}$$

holds for every element $f(z)$ of the space $\mathcal{L}(\phi)$. The space $\mathcal{L}(\phi)$ is contained contractively in $\mathcal{C}(z)$. The complementary space to $\mathcal{L}(\phi)$ in $\mathcal{C}(z)$ is a Herglotz space $\mathcal{L}(1 - \phi)$, which is contained contractively in $\mathcal{C}(z)$, such that the adjoint of multiplication by $W(z)$ in $\mathcal{C}(z)$ acts as a partially isometric transformation of $\mathcal{C}(z)$ onto the space $\mathcal{L}(1 - \phi)$. The polynomial elements of the space $\mathcal{L}(1 - \phi)$ are dense in the space. A power series $V(z)$, such that multiplication by $V(z)$ is an everywhere defined and contractive transformation in $\mathcal{C}(z)$, exists such that the adjoint of multiplication by $V(z)$ as a transformation in $\mathcal{C}(z)$ acts as a partially isometric transformation of $\mathcal{C}(z)$ onto the space $\mathcal{L}(1 - \phi)$ and such that the kernel of the adjoint transformation is invariant under multiplication by $z$. A power series $U(z)$ exists such that multiplication by $U(z)$ is an everywhere defined and contractive transformation in $\mathcal{C}(z)$, such that the adjoint of multiplication by $W(z)$ in $\mathcal{C}(z)$ is the composition of the adjoint of multiplication by $U(z)$ in $\mathcal{C}(z)$ and the adjoint of multiplication by $V(z)$ in $\mathcal{C}(z)$, and such that the range of the adjoint of multiplication by $U(z)$ in $\mathcal{C}(z)$ is orthogonal to the kernel of the adjoint of multiplication by $V(z)$ in $\mathcal{C}(z)$.

The Beurling factorization

$$W(z) = V(z)U(z)$$

results of a power series $W(z)$ such that multiplication by $W(z)$ is an everywhere defined and contractive transformation in $\mathcal{C}(z)$. The outer function $V(z)$ is a power series such that multiplication by $V(z)$ is an everywhere defined and contractive transformation in $\mathcal{C}(z)$ with range dense in $\mathcal{C}(z)$. The inner function $U(z)$ is a power series such that multiplication by $U(z)$ is a partially isometric transformation in $\mathcal{C}(z)$ and such that the range of multiplication by $U(z)$ in $\mathcal{C}(z)$ is orthogonal to the kernel of multiplication by $V(z)$.

The Nevanlinna factorization of a power series $W(z)$, such that multiplication by $W(z)$ is densely defined as a transformation in $\mathcal{C}(z)$, is a variant of the Beurling factorization. An outer function is again a power series $V(z)$ such that multiplication by $V(z)$ is an everywhere defined and contractive transformation in $\mathcal{C}(z)$ with range dense in $\mathcal{C}(z)$.

**Theorem 6.** If $W(z)$ is a power series such that multiplication by $W(z)$ is densely defined as a transformation in $\mathcal{C}(z)$, then an outer function $V(z)$ exists such that multiplication by

$$U(z) = W(z)V(z)$$

is an everywhere defined and contractive transformation in $\mathcal{C}(z)$ and such that no nonzero element $f(z)$ of the space $\mathcal{H}(V)$ exists such that $W(z)f(z)$ belongs to the space $\mathcal{H}(U)$.

**Proof of Theorem 6.** Multiplication by $W(z)$ is extended to a transformation with domain and range in ext $\mathcal{C}(z)$ which commutes with multiplication by $z$. An element $f(z)$ of the space ext $\mathcal{C}(z)$ belongs to the domain of multiplication by $W(z)$ as transformation in ext $\mathcal{C}(z)$ if $z^nf(z)$ belongs to the domain of multiplication by $W(z)$ in $\mathcal{C}(z)$ for some
nonnegative integer \( r \). Multiplication by \( W(z) \) in \( \text{ext } \mathcal{C}(z) \) takes \( z^r f(z) \) into \( z^r g(z) \) when multiplication by \( W(z) \) in \( \mathcal{C}(z) \) takes \( f(z) \) into \( g(z) \). The definition is independent of \( r \). Multiplication by \( W(z) \) in \( \text{ext } \mathcal{C}(z) \) is a densely defined transformation in \( \text{ext } \mathcal{C}(z) \), whose closure is however not assumed to be a transformation. The adjoint of multiplication by \( W(z) \) as a transformation in \( \text{ext } \mathcal{C}(z) \) is a transformation. The spectral subspace of contractivity for the adjoint of multiplication by \( W(z) \) in \( \text{ext } \mathcal{C}(z) \) and its orthogonal complement are invariant subspaces for multiplication and division by \( z \).

The space \( \text{ext } \mathcal{C}(z) \) is contained contractively in a Hilbert space \( \text{ext } \mathcal{P} \) such that a dense set of elements of the complementary space to \( \text{ext } \mathcal{C}(z) \) in \( \mathcal{P} \) belong to \( \text{ext } \mathcal{C}(z) \) and such that the adjoint of multiplication by \( W(z) \) as a transformation in \( \text{ext } \mathcal{C}(z) \) maps the intersection of its domain with the orthogonal complement of its spectral subspace of contractivity onto the intersection of \( \text{ext } \mathcal{C}(z) \) with the complementary space to \( \text{ext } \mathcal{C}(z) \) in \( \text{ext } \mathcal{P} \). The intersection of \( \text{ext } \mathcal{C}(z) \) with the complementary space to \( \text{ext } \mathcal{C}(z) \) in \( \mathcal{P} \) is invariant under division by \( z \). Division by \( z \) is an isometric transformation with respect to the scalar product of \( \mathcal{P} \) as well as with respect to the scalar product of the complementary space to \( \text{ext } \mathcal{C}(z) \) in \( \mathcal{P} \).

The canonical projection of \( \text{ext } \mathcal{C}(z) \) onto \( \mathcal{C}(z) \) determines a partially isometric transformation of \( \text{ext } \mathcal{P} \) onto a Hilbert space \( \mathcal{P} \), a dense set of whose elements belong to \( \mathcal{C}(z) \). The space \( \mathcal{C}(z) \) is contained contractively in the space \( \mathcal{P} \). The partially isometric transformation of \( \text{ext } \mathcal{P} \) onto \( \mathcal{P} \) acts as a partially isometric transformation of the complementary space to \( \text{ext } \mathcal{C}(z) \) in \( \mathcal{P} \) onto the complementary space to \( \mathcal{C}(z) \) in \( \mathcal{P} \). The intersection of \( \mathcal{C}(z) \) with \( \mathcal{P} \) and the intersection of \( \mathcal{C}(z) \) with the complementary space to \( \mathcal{C}(z) \) in \( \mathcal{P} \) are invariant subspaces for the difference-quotient transformation. The continuous extension of the difference-quotient transformation has an isometric adjoint in \( \mathcal{P} \) as well as in the complementary space to \( \mathcal{C}(z) \) in \( \mathcal{P} \).

Since the polynomial elements of \( \mathcal{C}(z) \) are dense in \( \mathcal{P} \), an isometric transformation of \( \mathcal{P} \) onto \( \mathcal{C}(z) \) exists which intertwines the continuous extension of the difference-quotient transformation in \( \mathcal{P} \) with the difference-quotient transformation in \( \mathcal{C}(z) \). Since \( \mathcal{C}(z) \) is contained contractively in \( \mathcal{P} \), a contractive transformation of \( \mathcal{C}(z) \) into itself is obtained which commutes with the difference-quotient transformation. The transformation is the adjoint of multiplication by \( V(z) \) for a power series \( V(z) \) such that multiplication by \( V(z) \) is everywhere defined and contractive as a transformation in \( \mathcal{C}(z) \). A Hilbert space \( \mathcal{H}(V) \) exists which is the state space of a canonical coisometric linear system with transfer function \( V(z) \). The Herglotz space \( \mathcal{L}(\phi) \) associated with the space \( \mathcal{H}(V) \) is contained contractively in \( \mathcal{C}(z) \). The continuous extension of the adjoint of multiplication by \( V(z) \) acts as an isometric transformation of \( \mathcal{P} \) onto \( \mathcal{C}(z) \). The adjoint of multiplication by \( V(z) \) as a transformation in \( \mathcal{C}(z) \) acts as an isometric transformation of \( \mathcal{C}(z) \) onto the complementary space \( \mathcal{L}(1 - \psi) \) to the space \( \mathcal{L}(\psi) \) in \( \mathcal{C}(z) \). The continuous extension of the adjoint of multiplication by \( V(z) \) as a transformation in \( \mathcal{C}(z) \) acts as an isometric transformation of the complementary space to \( \mathcal{C}(z) \) in \( \mathcal{P} \) onto the space \( \mathcal{L}(\psi) \).

A Hilbert space \( \text{ext } \mathcal{Q} \), which is contained contractively in the space \( \text{ext } \mathcal{P} \), exists such that the intersection of \( \text{ext } \mathcal{Q} \) with \( \text{ext } \mathcal{C}(z) \) is the range of the adjoint of multiplication by \( W(z) \) as a transformation in \( \text{ext } \mathcal{C}(z) \). The space \( \mathcal{Q} \) is the orthogonal sum of its
intersection with the spectral subspace of contractivity for multiplication by $W(z)$ in $\mathcal{C}(z)$ and the closure of its intersection with the orthogonal complement in $\text{ext} \mathcal{C}(z)$ of the spectral subspace. The complementary space to $\text{ext} \mathcal{C}(z)$ in $\text{ext} \mathcal{Q}$ is isometrically equal to the closure in $\text{ext} \mathcal{Q}$ of its intersection with the orthogonal complement of the spectral subspace. The adjoint of multiplication by $W(z)$ as a transformation in $\text{ext} \mathcal{C}(z)$ acts as a partially isometric transformation of its spectral subspace of contractivity onto the intersection of $\text{ext} \mathcal{Q}$ with the spectral subspace of contractivity for multiplication by $W(z)$ as a transformation in $\text{ext} \mathcal{C}(z)$. The space $\text{ext} \mathcal{Q}$ and its complementary space in the space $\text{ext} \mathcal{P}$ are invariant subspaces for the continuous extension of division by $z$. The continuous extension of division by $z$ is an isometric transformation in $\text{ext} \mathcal{Q}$ and its complementary space in $\text{ext} \mathcal{P}$.

The partially isometric transformation of $\text{ext} \mathcal{P}$ onto $\mathcal{P}$, which is determined by the canonical projection of $\text{ext} \mathcal{C}(z)$ onto $\mathcal{C}(z)$, acts as a partially isometric transformation of $\text{ext} \mathcal{Q}$ onto a Hilbert space which is contained contractively in $\mathcal{P}$. The canonical projection of $\text{ext} \mathcal{C}(z)$ onto $\mathcal{C}(z)$ acts as a partially isometric transformation of the complementary space to $\text{ext} \mathcal{Q}$ in $\text{ext} \mathcal{P}$ onto the complementary space to $\mathcal{Q}$ in $\mathcal{P}$. The space $\mathcal{Q}$ and its complementary space in $\mathcal{P}$ are invariant subspaces for the continuous extension of the difference-quotient transformation. The continuous extension of the difference-quotient transformation has an isometric adjoint in $\mathcal{Q}$ and in its complementary space in $\mathcal{P}$.

The power series

$$U(z) = W(z)V(z)$$

has properties which are derived from adjoints of multiplication transformations. Since multiplication by $W(z)$ is a densely defined transformation in $\mathcal{C}(z)$ by hypothesis, the adjoint of multiplication by $W(z)$ as a transformation in $\mathcal{C}(z)$ is contained in the closure of the composition of the adjoint of multiplication by $W(z)$ as a transformation in $\text{ext} \mathcal{C}(z)$ with the canonical projection of $\text{ext} \mathcal{C}(z)$ onto $\mathcal{C}(z)$. The range of the adjoint of multiplication by $W(z)$ as a transformation in $\mathcal{C}(z)$ is contained in $\mathcal{Q}$. The continuous extension of the adjoint of multiplication by $W(z)$ as a transformation in $\mathcal{C}(z)$ is a contractive transformation of $\mathcal{C}(z)$ into $\mathcal{Q}$. The continuous extension of the adjoint of multiplication by $V(z)$ as a transformation in $\mathcal{C}(z)$ acts as a contractive transformation of $\mathcal{Q}$ into $\mathcal{C}(z)$. Since the adjoint of multiplication by $U(z)$ in $\mathcal{C}(z)$ is contained in the composition of the continuous extension of the adjoint of multiplicative by $W(z)$ in $\mathcal{C}(z)$ with the continuous extension of the adjoint of multiplication by $V(z)$ in $\mathcal{C}(z)$ and since $\mathcal{C}(z)$ is contained contractively in $\mathcal{Q}$, the adjoint of multiplication by $U(z)$ is an everywhere defined and contractive transformation in $\mathcal{C}(z)$. The contractive property is first verified on polynomial elements of $\mathcal{C}(z)$. If then follows for all elements of $\mathcal{C}(z)$. Multiplication by $U(z)$ is everywhere defined and contractive as a transformation in $\mathcal{C}(z)$.

A Hilbert space $\mathcal{H}(U)$ exists which is the state space of a canonical coisometric linear system with transfer function $U(z)$. The adjoint of multiplication by $U(z)$ as a transformation in $\mathcal{C}(z)$ acts as a partially isometric transformation of $\mathcal{C}(z)$ onto the complementary space $\mathcal{L}(1 - \phi)$ in $\mathcal{C}(z)$ to the Herglotz space $\mathcal{L}(\phi)$ associated with the space $\mathcal{H}(0)$.

Since the complementary space to $\mathcal{C}(z)$ in the space $\mathcal{P}$ is contained isometrically in the space $\mathcal{Q}$, no nonzero element of the complementary space to $\mathcal{C}(z)$ in the space $\mathcal{P}$ belongs
to the complementary space to the space $Q$ in the space $P$. Since the continuous extension of the adjoint of multiplication by $V(z)$ in $C(z)$ acts as an isometric transformation of the complementary space to $C(z)$ in $P$ onto the space $L(\psi)$ and of the complementary space to $Q$ in $P$ onto $L(\phi)$, the intersection of the spaces $L(\phi)$ and $L(\psi)$ contains no nonzero element.

The space $H(V)$ is the set of elements $f(z)$ of $C(z)$ such that the adjoint of multiplication by $V(z)$ in $C(z)$ maps $f(z)$ into an element $g(z)$ of the space $L(\psi)$. The identity

$$\|f(z)\|_{H(V)}^2 = \|f(z)\|^2_{C(z)} + \|g(z)\|^2_{L(\psi)}$$

is then satisfied. Since the continuous extension of the adjoint of multiplication by $V(z)$ as a transformation in $C(z)$ acts as an isometric transformation of the complementary space to $C(z)$ in the space $P$ onto the space $L(\psi)$, the space $H(V)$ is the intersection of $C(z)$ with the complementary space to $C(z)$ in the space $P$. The square of the norm of an element of the space $H(V)$ is the sum of the square of its norm as an element of $C(z)$ and the square of its norm as an element of the complementary space to $C(z)$ in the space $P$.

The space $H(U)$ is the set of element $f(z)$ of $C(z)$ such that the adjoint of multiplication by $U(z)$ in $C(z)$ maps $f(z)$ into an element $g(z)$ of the space $L(\phi)$. The identity

$$\|f(z)\|_{H(U)}^2 = \|f(z)\|^2_{C(z)} + \|g(z)\|^2_{L(\phi)}$$

is then satisfied. Since the continuous extension of the adjoint of multiplication by $V(z)$ as a transformation in $C(z)$ acts as an isometric transformation of the complementary space to $Q$ in $P$ onto the space $L(\phi)$, the space $H(U)$ is the set of elements $f(z)$ of $C(z)$ such that the adjoint of multiplication by $W(z)$ in $C(z)$ maps $f(z)$ into an element $h(z)$ of the complementary space to $Q$ in $P$. Since $h(z)$ then belongs to the spectral subspace of contractivity for multiplication by $W(z)$ as a transformation in ext $C(z)$, the element $f(z)$ of $C(z)$ is the projection of an element of ext $C(z)$ which belongs to the spectral subspace of contractivity for the adjoint of multiplication by $W(z)$ as a transformation in ext $C(z)$.

If $f(z)$ is an element of the space $H(V)$ such that $W(z)f(z)$ belongs to the space $H(U)$, then $W(z)f(z)$ belongs to the orthogonal complement of the spectral subspace of contractivity for multiplication by $W(z)$ as a transformation in ext $C(z)$ since $f(z)$ belongs to the orthogonal complement of the spectral subspace of contractivity for multiplication by $W(z)$ as a transformation in ext $C(z)$. An element $g(z)$ of the spectral subspace of contractivity for the adjoint of multiplication by $W(z)$ as a transformation in ext $C(z)$ exists which has $W(z)f(z)$ as its orthogonal projection in $C(z)$. The product $W(z)f(z)$ is equal to zero since it is orthogonal to $g(z)$. The element $f(z)$ of $C(z)$ is equal to zero since it is orthogonal to the spectral subspace of contractivity for multiplication by $W(z)$ as a transformation in ext $C(z)$.

This completes the proof of the theorem.

The Hilbert space $H(U)$ is contained continuously and isometrically in a Krein space $H(W)$, whose elements are power series, such that multiplication by $W(z)$ acts as an
isometric transformation of the anti-space of the Hilbert space \( \mathcal{H}(V) \) onto the orthogonal complement of the space \( \mathcal{H}(U) \) in the space \( \mathcal{H}(W) \). The space \( \mathcal{H}(U') \) corresponding to the power series \( U(z) = zU(z) \) with complex coefficients is the set of power series \( f(z) \) with complex coefficients such that \( [f(z) - f(0)]/z \) belongs to the space \( \mathcal{H}(U) \). The identity for difference quotients

\[
\|([f(z) - f(0)]/z)^2 \|_{\mathcal{H}(U)} = \|f(z)^2\|_{\mathcal{H}(U')} - f(0)^2 f(0)
\]

is satisfied. The space \( \mathcal{H}(U) \) is contained contractively in the space \( \mathcal{H}(U') \). The space \( \mathcal{H}(W') \) corresponding to the power series \( W(z) = zW(z) \) with complex coefficients is the set of power series \( f(z) \) with complex coefficients such that \( [f(z) - f(0)]/z \) belongs to the space \( \mathcal{H}(W) \). The identity for difference quotients

\[
\langle [f(z) - f(0)]/z, [f(z) - f(0)]/z \rangle_{\mathcal{H}(W)} = \langle f(z), f(z) \rangle_{\mathcal{H}(W')} - f(0)^2 f(0)
\]

is satisfied. The space \( \mathcal{H}(U') \) is contained continuously and isometrically in the space \( \mathcal{H}(W') \). Multiplication by \( W(z) \) is an isometric transformation of the anti-space of the Hilbert space \( \mathcal{H}(V) \) onto the orthogonal complement of the space \( \mathcal{H}(U') \) in the space \( \mathcal{H}(W') \). The space \( \mathcal{H}(W) \) is contained continuously and contractively in the space \( \mathcal{H}(W') \). Multiplication by \( W(z) \) is a partially isometric transformation of the space of complex numbers onto the complementary space to the space \( \mathcal{H}(W) \) in the space \( \mathcal{H}(W') \). The space \( \mathcal{H}(W) \) is the state space of a canonical coisometric linear system with transfer function \( W(z) \).

A canonical unitary linear is constructed from a canonical coisometric linear system with state space \( \mathcal{H}(W) \) and transfer function \( W(z) \) when the inequality for difference quotients

\[
\langle [f(z) - f(0)]/z, [f(z) - f(0)]/z \rangle_{\mathcal{H}(W)} \leq \langle f(z), f(z) \rangle_{\mathcal{H}(W')} - f(0)^2 f(0)
\]

holds for every element \( f(z) \) of the space. The inequality is satisfied by the canonical coisometric linear system constructed when multiplication by \( W(z) \) is densely defined as a transformation in \( \mathcal{C}(z) \). The elements of the state space \( \mathcal{D}(W) \) of the canonical unitary linear system are pairs \( (f(z), g(z)) \) of power series. Power series \( f(z) \) and \( g(z) = \sum a_n z^n \)

with complex coefficients determine an element of the space \( \mathcal{D}(W) \) if \( f(z) \) is an element of the space \( \mathcal{H}(W) \) such that

\[
z^{r+1} f(z) - W(z)(a_0 z^r + \ldots + a_r)
\]
belongs to the space $\mathcal{H}(W)$ for every nonnegative integer $r$ and such that the sequence of numbers
\[
\langle z^{r+1} f(z) - W(z)(a_0 z^r + \ldots + a_r), z^{r+1} f(z) - W(z)(a_0 z^r + \ldots + a_r) \rangle_{\mathcal{H}(W)} + a_0 \bar{a}_0 + \ldots + a_r \bar{a}_r
\]
is bounded. The inequality for difference quotients in the space $\mathcal{H}(W)$ implies that the sequence is nondecreasing. The limit of the sequence is taken as the definition of the scalar self–product
\[
\langle f(z), g(z) \rangle, \langle f(z), g(z) \rangle_{\mathcal{D}(W)}.
\]
The space $\mathcal{D}(W)$ is a Krein space. A contractive partially isometric transformation of the space $\mathcal{D}(W)$ onto the space $\mathcal{H}(W)$ is defined by taking $(f(z), g(z))$ into $f(z)$. A continuous transformation of the space $\mathcal{D}(W)$ into itself is defined by taking $(f(z), g(z))$ into
\[
([f(z) - f(0)]/z, zg(z) - W^*(z)f(0)).
\]
The identity for difference quotients
\[
\langle ([f(z) - f(0)]/z, zg(z) - W^*(z)f(0)), ([f(z) - f(0)]/z, zg(z) - W^*(z)f(z)) \rangle_{\mathcal{D}(W)}
\]
is satisfied. The adjoint transformation of the space $\mathcal{D}(W)$ into itself takes $(f(z), g(z))$ into
\[
(zf(z) - W(z)g(0), [g(z) - g(0)]/z).
\]
The identity for difference quotients
\[
\langle (zf(z) - W(z)g(0), [g(z) - g(0)]/z), (zf(z) - W(z)g(0), [g(z) - g(0)]/z) \rangle_{\mathcal{D}(W)}
\]
is satisfied.

A construction has been made of the state space $\mathcal{D}(W)$ of a canonical unitary linear system with transfer function $W(z)$. The main transformation takes $(f(z), g(z))$ into
\[
([f(z) - f(0)]/z, zg(z) - W^*(z)f(0)).
\]
The input transformation takes $c$ into
\[
([W(z) - W(0)]c/z, [1 - W^*(z)W(0)]c).
\]
The output transformation takes $(f(z), g(z))$ into $f(0)$. The external operator is $W(0)$. The unitary property of the linear system is a consequence of the two identities for difference quotients. The transformation which takes $(f(z), g(z))$ into $(g(z), f(z))$ maps the space $\mathcal{D}(W)$ isometrically onto the state space $\mathcal{D}(W^*)$ of a canonical unitary linear system with transfer function $W^*(z)$.

Uniqueness of a canonical unitary linear system with transfer function $W(z)$ is obtained when multiplication by $W(z)$ is densely defined as a transformation in $\mathcal{C}(z)$. 
Theorem 7. If $W(z)$ is a power series such that multiplication by $W(z)$ is densely defined as a transformation in $C(z)$, if $V(z)$ and

$$U(z) = W(z)V(z)$$

are power series such that multiplication by $U(z)$ and multiplication by $V(z)$ are everywhere defined and contractive as transformations in $C(z)$, if no nonzero element $f(z)$ of the space $H(V)$ exists such that $W(z)f(z)$ belongs to the space $H(U)$, and if $D(W)$ is the state space of a canonical unitary linear system with transfer function $W(z)$, then a contractive partially isometric transformation of the space $D(W)$ onto a Krein space $H(W)$, which is the state space of a canonical coisometric linear system with transfer function $W(z)$, is defined by taking $(f(z); g(z))$ into $f(z)$. The space $H(U)$ is contained continuously and isometrically in the space $H(W)$. Multiplication by $W(z)$ is a partially isometric transformation of the anti-space of the space $H(V)$ onto the orthogonal complement of the space $H(U)$ in the space $H(W)$.

Proof of Theorem 7. A transformation of the Cartesian product of the space $D(W)$ and the space $D(V)$ onto a vector space $D$, whose elements are pairs of power series with complex coefficients, is defined by taking an element $(f(z), g(z))$ of the space $D(W)$ and an element $(h(z), k(z))$ of the space $D(V)$ into the element $(u(z), v(z))$ of the space $D$ defined by

$$u(z) = f(z) + W(z)h(z)$$

and

$$v(z) = k(z) + V^*(z)g(z).$$

The space $D$ is the state space of a linear system with transfer function $U(z)$. The main transformation takes $(u(z), v(z))$ into

$$([u(z) - u(0)]/z, zv(z) - U^*(z)u(0)).$$

If

$$u(z) = f(z) + W(z)h(z)$$

and

$$v(z) = k(z) + V^*(z)g(z),$$

then

$$[u(z) - u(0)]/z = f'(z) + W(z)h'(z)$$

and

$$zv(z) - U^*(z)u(0) = k'(z) + V^*(z)g'(z)$$

with $(f'(z), g'(z))$ the element of the space $D(W)$ defined by

$$f'(z) = [f(z) - f(0)]/z + [W(z) - W(0)]h(0)/z$$

and

$$g'(z) = zg(z) - W^*(z)h(0) + [1 - W^*(z)W(0)]h(0).$$
and with \((h'(z), k'(z))\) the element of the space \(D(V)\) defined by

\[
h'(z) = \frac{h(z) - h(0)}{z}
\]

and

\[
k'(z) = zk(z) - V^*(z)h(0).
\]

The input transformation takes a complex number \(c\) into

\[
([1 - U(z)U(0)^{-}]c, [U^*(z) - U^*(0)]c/z)
\]

where

\[
[1 - U(z)U(0)^{-}]c = [1 - W(z)W(0)^{-}]c + W(z)[1 - V(z)V(0)^{-}]W(0)^{-}c
\]

and

\[
[U^*(z) - U^*(0)]c/z = [V^*(z) - V^*(0)]W^*(0)c/z + V^*(z)[U^*(z) - U^*(0)]c/z
\]

with

\[
([1 - W(z)W(0)^{-}]c, [W^*(z) - W^*(0)]c/z)
\]

an element of the space \(D(W)\) and

\[
([1 - V(z)V(0)^{-}]W(0)^{-}c, [V^*(z) - V^*(0)]W^*(0)c/z)
\]

an element of the space \(D(W)\). The output transformation takes \((u(z), v(z))\) into \(u(0)\). If

\[
u(z) = f(z) + W(z)h(z)
\]

and

\[
v(z) = k(z) + V^*(z)g(z)
\]

with \((f(z), g(z))\) in the space \(D(W)\) and \((h(z), k(z))\) in the space \(D(V)\), then

\[
u(0) = f(0) + W(0)g(0).
\]

The external operator is

\[
U(0) = W(0)V(0).
\]

The matrix entries of the linear system with state space \(D\) and transfer function \(U(z)\) are constructed from the matrix entries of a unitary linear system whose state space is the Cartesian product of the space \(D(W)\) and the space \(D(V)\) and whose transfer function is \(U(z)\). A partially isometric transformation exists of the Cartesian product of the spaces \(D(W)\) and \(D(V)\) onto the space \(D(U)\) which is compatible with the structure of these spaces as state spaces of unitary linear systems with transfer function \(U(z)\). Since the transformation of the Cartesian product space onto the space \(D\) is identical with the transformation of the Cartesian product space onto the space \(D(U)\), the space \(D\) is a
Hilbert space which is the state space \( \mathcal{D}(U) \) of a canonical unitary linear system with transfer function \( U(z) \). The transformation of the Cartesian product space onto the space \( \mathcal{D} \), equal to \( \mathcal{D}(U) \), is a partial isometry.

A Krein space \( \mathcal{E} \) is constructed whose elements are the pairs \((f(z), g(z))\) of power series such that
\[
(-f(z), V^*(z)g(z))
\]
belongs to the space \( \mathcal{D}(V) \) and
\[
(W(z)f(z), -g(z))
\]
belongs to the space \( \mathcal{D}(W) \). The scalar product is defined in the space so that the identity
\[
\langle (f(z), g(z)), (f(z), g(z)) \rangle_\mathcal{E} = \langle (-f(z), V^*(z)g(z)), (-f(z), V^*(z)g(z)) \rangle_{\mathcal{D}(V)}
\]
\[+(W(z)f(z), -g(z)), (W(z)f(z), -g(z)) \rangle_{\mathcal{D}(W)}
\]
is satisfied. An isometric transformation of the space \( \mathcal{E} \) onto itself is defined by taking \((f(z), g(z))\) into
\[
([f(z) - f(0)]/z, f(0) + zg(z)).
\]
The inverse isometric transformation takes \((f(z), g(z))\) into
\[
(g(0) + zf(z), [g(z) - g(0)]/z).
\]

An inverse isometric transformation exists of the space \( \mathcal{D}(U) \) into the Cartesian product of the spaces \( \mathcal{D}(W) \) and the space \( \mathcal{D}(V) \). Every element of the space \( \mathcal{D}(U) \) is uniquely of the form
\[
(f(z) + W(z)h(z), k(z) + V^*(z)g(z))
\]
for elements \((f(z), g(z))\) of the space \( \mathcal{D}(W) \) and \((h(z), k(z))\) of the space \( \mathcal{D}(V) \) such that the identity
\[
\langle (f(z), g(z)), (W(z)u(z), -v(z)) \rangle_{\mathcal{D}(W)} = \langle (h(z), k(z)), (u(z), -V^*(z)v(z)) \rangle_{\mathcal{D}(V)}
\]
holds for every element \((u(z), v(z))\) of the space \( \mathcal{E} \). The image of the space \( \mathcal{E} \) in the Cartesian product of the space \( \mathcal{D}(W) \) and the space \( \mathcal{D}(V) \) consists of the pairs of elements \((W(z)u(z), -v(z))\) of the space \( \mathcal{D}(W) \) and elements \((-u(z), V^*(z)v(z))\) of the space \( \mathcal{D}(V) \) which are parametrized by elements \((u(z), v(z))\) of the space \( \mathcal{E} \). The pair of elements \((f(z), g(z))\) of the space \( \mathcal{D}(W) \) and \((h(z), k(z))\) of the space \( \mathcal{D}(V) \) is orthogonal in the Cartesian product space to the image of the space \( \mathcal{E} \).

If \((h(z), k(z))\) is an element of the space \( \mathcal{D}(V) \), an element \((u(z), v(z))\) of the space \( \mathcal{E} \) exists such that an element of the Cartesian product space in the image of \( \mathcal{D}(U) \) is obtained as the pair consisting of the element \((W(z)u(z), -v(z))\) of the space \( \mathcal{D}(W) \) and the element
\[
(h(z) - u(z), k(z) + V^*(z)v(z))
\]
of the space \( \mathcal{D}(V) \). Then
\[
(W(z)[h(z) - u(z)], k(z) + V^*(z)v(z))
\]
is an element of the space $\mathcal{D}(U)$. Since no nonzero element $f(z)$ of the space $\mathcal{H}(V)$ exists such that $W(z)f(z)$ belongs to the space $\mathcal{H}(U)$, no nonzero element $(f(z), g(z))$ of the space $\mathcal{D}(V)$ exists such that

$$(W(z)f(z), g(z))$$

belong to the space $\mathcal{D}(U)$. It follows that

$$(h(z), k(z)) = (u(z), -V^*(z)v(z)).$$

The image of the space $\mathcal{D}(U)$ in the Cartesian product space consists of pairs of elements $(f(z), g(z))$ of the space $\mathcal{D}(W)$ such that $(f(z), V^*(z)g(z))$ belongs to the space $\mathcal{D}(U)$ and the zero element of the space $\mathcal{D}(V)$. A continuous isometric transformation of the space $\mathcal{D}(U)$ into the space $\mathcal{D}(W)$ is defined by taking $(f(z), V^*(z)g(z))$ into $(f(z), g(z))$. The orthogonal complement in the space $\mathcal{D}(W)$ of the image of the space $\mathcal{D}(U)$ consists of the elements of the space $\mathcal{D}(W)$ of the form $(W(z)u(z), -v(z))$ with $(u(z), v(z))$ in the space $\mathcal{E}$. Since $(-u(z), V^*(z)v(z))$ then belongs to the space $\mathcal{D}(V)$, the identity

$$-\langle (W(z)u(z), -v(z)), (W(z)u(z), -v(z)) \rangle_{\mathcal{D}(W)}$$

$$= \langle (-u(z), V^*(z)v(z)), (-u(z), V^*(z)v(z)) \rangle_{\mathcal{D}(V)}$$

is then satisfied. An isometric transformation of the anti–space of the space $\mathcal{D}(V)$ onto the orthogonal complement in the space $\mathcal{D}(W)$ of the image of the space $\mathcal{D}(U)$ is defined by taking $(u(z), -V^*(z)v(z))$ into $(W(z)u(z), -v(z))$.

The space $\mathcal{D}(W)$ is isometrically equal to the state space of the canonical unitary linear system with transfer function $W(z)$ which is constructed from the state space $\mathcal{H}(W)$ of the canonical coisometric linear system with transfer function $W(z)$ when multiplication by $W(z)$ is densely defined as a transformation in $\mathcal{C}(z)$. A contractive partially isometric transformation of the space $\mathcal{D}(W)$ onto the space $\mathcal{H}(W)$ is defined by taking $(f(z), g(z))$ into $f(z)$.

This completes the proof of the theorem.

The factorization of power series which represent analytic functions is treated when the Nevanlinna factorization is not assumed applicable. Power series are multiplied when they are the transfer functions of canonical coisometric linear systems which are related to each other by a Herglotz space.

Assume that $\mathcal{H}(U)$ is the state space of a contractive coisometric linear system with transfer function $U(z)$, that $\mathcal{H}(V)$ is the state space of a canonical coisometric linear system with transfer function $V(z)$, and that a Herglotz space $\mathcal{L}(\phi)$ exists whose elements are elements $f(z)$ of the space $\mathcal{H}(V)$ such that $U(z)f(z)$ belongs to the space $\mathcal{H}(U)$ and which satisfies the identity

$$\langle f(z), f(z) \rangle_{\mathcal{L}(\phi)} = \langle f(z), f(z) \rangle_{\mathcal{H}(V)} + \langle U(z)f(z), U(z)f(z) \rangle_{\mathcal{H}(U)}$$

for every element $f(z)$ of the space. Then the set of sums

$$h(z) = f(z) + U(z)g(z)$$
with $f(z)$ in the space $H(U)$ and $g(z)$ in the space $H(V)$ is a Krein space $H(W)$ whose scalar product is determined by the greatest lower bound

$$\langle h(z), h(z) \rangle_{H(W)} = \inf \{ \langle f(z), f(z) \rangle_{H(U)} + \langle g(z), g(z) \rangle_{H(V)} \}$$

taken over all such representations of $h(z)$. The Krein space $H(W)$ is the state space of a contractive canonical coisometric linear system with transfer function $W(z) = U(z)V(z)$.

If the domain of multiplication by $W(z)$ in $C(z)$ contains a nonzero element for a power series $W(z)$, then the closure of the domain of multiplication by $W(z)$ is the range of multiplication by $S(z)$ for a power series $S(z)$ with nonzero constant coefficient such that multiplication by $S(z)$ is an everywhere defined and isometric transformation in $C(z)$. Since multiplication by

$$W'(z) + W(z)S(z)$$

is a densely defined transformation in $C(z)$, a Krein space exists which is the state space $H(W')$ of a contractive canonical coisometric linear system with transfer function $W'(z)$. Power series $U(z)$ and $V'(z)$ exist such that multiplication by $U(z)$ and $V'(z)$ are everywhere defined and contractive transformations in $C(z)$, such that the space $H(U)$ is contained continuously and isometrically in the space $H(W')$, and such that multiplication by $W'(z)$ is an isometric transformation of the anti–space of the space $H(V')$ onto the orthogonal complement of the space $H(U)$ in the space $H(W')$. The identity

$$U(z) = W'(z)V'(z)$$

is satisfied. A space $H(W)$ exists which is the state space of a contractive canonical coisometric linear system with transfer function $W(z)$. The space $H(W)$ is contained continuously and isometrically in the space $H(W')$. Multiplication by $W(z)$ is an isometric transformation of the anti–space of the space $H(S)$ onto the orthogonal complement of the space $H(W)$ in the space $H(W')$. Multiplication by

$$V(z) = S(z)V'(z)$$

is an everywhere defined and contractive transformation in $C(z)$. The space $H(S)$ is contained isometrically in the space $H(V)$. Multiplication by $W(z)$ is an isometric transformation of the anti–space of the space $H(V)$ onto the orthogonal complement of the space $H(U)$ in the space $H(W)$.

If the state space $H(W)$ of a contractive canonical coisometric linear system with transfer function $W(z)$ is a Krein space of finite dimension, then the domain of multiplication by $W(z)$ as a transformation in $C(z)$ contains no nonzero element. The identity

$$U(z) = W(z)V(z)$$

holds for power series $U(z)$ and $V(z)$ such that multiplication by $U(z)$ and $V(z)$ are everywhere defined and contractive transformations in $C(z)$, such that the space $H(U)$ is
contained continuous and isometrically in the space $\mathcal{H}(W)$, and such that multiplication by $W(z)$ is an isometric transformation of the anti-space of the space $\mathcal{H}(V)$ onto the orthogonal complement of the space $\mathcal{H}(U)$ in the space $\mathcal{H}(W)$. Since the spaces $\mathcal{H}(U)$ and $\mathcal{H}(V)$ have finite dimension, the power series $U(z)$ and $V(z)$ are finite Blaschke products. Each power series is determined within a constant factor of absolute value one by the zeros of the represented function. A zero $\omega$ in the unit disk contributes a factor

$$(z - w)/(1 - w^{-} z)$$

to the product. Since the space $\mathcal{H}(W)$ admits an orthogonal decomposition in terms of the spaces $\mathcal{H}(U)$ and $\mathcal{H}(V)$, the Blaschke products for $U(z)$ and $V(z)$ have no common zeros.

The convergence condition for a Blaschke product with an infinite sequence of zeros $\omega_n$ in the unit disk is finiteness of the sum

$$\sum (1 - w_n^{-} w_n).$$

The product

$$\prod (z - w_n)/(1 - w_n^{-} z)$$

then converges in the metric topology of $C(z)$ and represents a power series $W(z)$ such that multiplication by $W(z)$ is everywhere defined and isometric in $C(z)$. The state space $\mathcal{H}(W)$ of the canonical coisometric linear system with transfer function $W(z)$ is then a Hilbert space which is contained isometrically in $C(z)$.

The estimation theory for Riemann mapping functions is a variant of the theory of canonical coisometric linear systems. The Hilbert space $C(z)$ of square summable power series is replaced by Dirichlet Hilbert space $\mathcal{G}$. The elements of the space are equivalence classes of power series

$$f(z) = \sum a_n z^n$$

such that the sum

$$\|f(z)\|_{\mathcal{G}}^2 = \sum n|a_n|^2$$

is finite. Power series are equivalent as elements of the space if they differ by a constant.

The square of the Dirichlet norm is computable as an area integral

$$\pi \|f(z)\|_{\mathcal{G}}^2 = \iint |f'(z)|^2 dx dy$$

taken over the unit disk. The integrand is a Jacobian determinant

$$|f'(z)|^2 = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$
of the real and imaginary parts of

\[ f(z) = u(z) + iv(z) \]

in view of the Cauchy–Riemann equations

\[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}
\]

and

\[
\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.
\]

If a power series \( W(z) \) with constant coefficient zero represents an injective mapping of the disk into itself, then

\[ g(z) = f(W(z)) \]

is an element of the Dirichlet Hilbert space whenever \( f(z) \) is an element of the Dirichlet Hilbert space since

\[
\|g(z)\|_{\mathcal{G}}^2 = \iint |f'(z)|^2 dx dy
\]

with integration over the region onto which \( W(z) \) maps the unit disk. The inequality

\[
\|g(z)\|_{\mathcal{G}} \leq \|f(z)\|_{\mathcal{G}}
\]

is satisfied.

The extension space of the Dirichlet Hilbert space is analogous to the extension space \( \text{ext} \mathcal{C}(z) \) of the space \( \mathcal{C}(z) \) of square summable power series. The Dirichlet Krein space \( \text{ext} \mathcal{G} \) is the set of equivalence classes of Laurent series

\[ f(z) = \sum a_n z^n \]

such that the sum

\[
\langle f(z), f(z) \rangle = \sum n |a_n|^2
\]

is absolutely convergent. Laurent series are considered equivalent if they differ by a constant. The Dirichlet Hilbert space is contained continuously and isometrically in the Dirichlet Krein space. The transformation which takes \( f(z) \) into \( f(z^{-1}) \) is an isometry of the Dirichlet Hilbert space onto the anti-space of its orthogonal complement in the Dirichlet Krein space. A computation of the scalar self–product is made for a dense set of elements of the Dirichlet Krein space. These elements have only a finite number of nonzero coefficients for negative powers of \( z \). Such a Laurent series represents a function whose product by a power of \( z \) is analytic in the unit disk. The scalar self–product in the Dirichlet Krein space is formally represented by a Stieltjes integral

\[
2\pi i \langle f(z), f(z) \rangle_{\text{ext} \mathcal{G}} = \int f(z)^{-1} df(z)
\]
over the boundary of the unit disk. The integral is interpreted as a limit of integrals over simple closed curves in the unit disk which converge to the boundary. It is sufficient to apply circles with center at the origin and with radius less than one.

The Grunsky Hilbert space $\mathcal{G}(W)$ is defined when $W(z)$ is a power series with constant coefficient zero which represents an injective mapping of the unit disk into itself. The range of the contractive transformation of the Dirichlet Hilbert space $\mathcal{G}$ into itself which takes $f(z)$ into $f(W(z))$ is a Hilbert space which is contained contractively in the space $\mathcal{G}$ when considered with the scalar product such that the transformation is a partial isometry. The space $\mathcal{G}(W)$ is defined as the complementary space in $\mathcal{G}$ of the range of the transformation. The elements of the space are represented by power series with constant coefficient zero. The function
\[
\log \frac{1 - W(z)W(w)}{1 - zw}
\]
of $z$ is represented by an element of the space $\mathcal{G}(W)$ when $w$ belongs to the unit disk and acts as reproducing kernel function for function values at $w$. If $f(z)$ belongs to the space $\mathcal{G}(W)$, then
\[
\exp f(z)
\]
belongs to the state space $\mathcal{H}(W)$ of the canonical coisometric linear system with transfer function $W(z)$. The inequality
\[
\|\exp f(z)\|_{\mathcal{H}(W)}^2 \leq \exp \|g(z)\|_{\mathcal{G}(W)}^2
\]
is satisfied. The identity
\[
\langle \exp f(z), \exp g(z) \rangle_{\mathcal{H}(W)} = \exp \langle f(z), g(z) \rangle_{\mathcal{G}(W)}
\]
holds for every element $g(z)$ of the space $\mathcal{G}(W)$ if equality holds.

A power series $W(z)$ with constant coefficient zero which represents an injective mapping of the unit disk into itself determines a contractive transformation $f(z)$ into $f(W(z))$ of the Dirichlet Krein space into itself. It is sufficient to verify the contractive property for a dense set of elements of the space. If an element $f(z)$ of the space has only a finite number of nonzero coefficients for negative powers of $z$, then the identity
\[
2\pi i \langle f(z), f(z) \rangle_{\text{ext } \mathcal{G}} = \int f(z)^{-1} df(z)
\]
holds with integration over the unit circle. The integral is interpreted as a limit as $t$ increases to one of integrals over circles of radius $t$ about the origin. The identity
\[
2\pi i \langle f(W(z)), f(W(z)) \rangle_{\text{ext } \mathcal{G}} = \int f(z)^{-1} df(z)
\]
holds with integration over the boundary of the region onto which $W(z)$ maps the unit disk. The integral is interpreted as a limit of integrals over the boundary of the regions onto which $W(z)$ maps disk of radius less than one about the origin. The identity
\[
\pi \langle f(z), f(z) \rangle_{\text{ext } \mathcal{G}} - \pi \langle f(W(z)), f(W(z)) \rangle_{\text{ext } \mathcal{G}} = \int |f'(z)|^2 dx dy
\]
holds with integration over the complement in the unit disk of the region onto which $W(z)$ maps the unit disk.

A continuous and contractive transformation has been constructed of the Dirichlet Krein space into itself which takes $f(z)$ into $f(W(z))$ for every element $f(z)$ of the space. The transformation has an everywhere defined and contractive adjoint. If a power series $W(z)$ with constant coefficient zero represents an injective mapping of the unit disk into itself, then the conjugate power series $W^*(z)$ has constant coefficient zero and represents an injective mapping of the unit disk into itself. The adjoint transformation takes $g(W^*(z^{-1}))$ into $g(z^{-1})$ for every element $g(z)$ of the Dirichlet Krein space. The identity

$$\langle f(z), g(z^{-1}) \rangle_{\text{ext} \mathcal{G}} = \langle f(W(z)), g(W^*(z^{-1})) \rangle_{\text{ext} \mathcal{G}}$$

holds for all elements $f(z)$ and $g(z)$ of the Dirichlet Krein space. It is sufficient to verify the identity when $f(z)$ and $g(z)$ have only a finite number of nonzero coefficients for negative powers of $z$. The identity

$$2\pi i \langle f(z), g(z^{-1}) \rangle_{\text{ext} \mathcal{G}} = \int g^*(z) df(z)$$

holds formally with integration over the unit circle. The integral is interpreted as a limit of integrals over circles of radius less than one about the origin. The identity

$$2\pi i \langle f(W(z)), g(W^*(z^{-1})) \rangle_{\text{ext} \mathcal{G}} = \int g^*(z) df(z)$$

holds formally with integration over the boundary of the region onto which $W(z)$ maps the unit disk. The integral is interpreted as a limit of integrals over the boundary of the region onto which $W(z)$ maps disks of radius less than one about the origin. The identity is an application of the Cauchy formula in the complement of the unit disk of the region onto which maps the unit disk.

The Grunsky Hilbert space $\mathcal{G}(W)$ is defined when a power series $W(z)$ with constant coefficient zero represents an injective mapping of the unit disk into itself. A continuous and contractive transformation of the Dirichlet Krein space into itself exists which takes $f(z)$ into $f(W(z))$ whenever an element $f(z)$ of the space has only a finite number of nonzero coefficients for negative powers of $z$. The range of the transformation is a Krein space which is contained continuously and contractively in the Dirichlet Krein space when considered with the scalar product such that the transformation is a partial isometry. Since the transformation has a contractive adjoint in the Dirichlet Krein space, the complementary space to the range in the Dirichlet Krein space is a Hilbert space whose elements are well-defined Laurent series. The Grunsky Hilbert space $\mathcal{G}(W)$, which is the extension space of the Grunsky Hilbert space $\mathcal{G}(W)$, is the set of pairs $(f(z), g(z))$ of power series such that $f(z)$ has constant coefficient zero and such that

$$f(z) + g(z^{-1})$$

belongs to the complementary space. The scalar product in the space $\text{ext} \mathcal{G}(W)$ is defined so that the canonical transformation onto the complementary space is an isometry. A partially
isometric transformation of the Grunsky Hilbert space \( \mathcal{G}(W) \) onto the Grunsky Hilbert space \( \mathcal{G}(W) \) is defined by taking \( (f(z), g(z)) \) into \( f(z) \). An isometric transformation of the Grunsky Hilbert space \( \mathcal{G}(W) \) onto the Grunsky Hilbert space \( \mathcal{G}(W^*) \) is defined by taking \( (f(z), g(z)) \) into
\[
(g(z) - g(0), g(0) + f(z)).
\]
A continuous linear functional on the Grunsky Hilbert space \( \mathcal{G}(W) \) is defined by taking \( (f(z), g(z)) \) into \( f(w) \) whenever \( w \) belongs to the unit disk. The reproducing kernel function for the linear functional is the element
\[
\left( \log \frac{1 - W(z)W(w)^{-1}}{1 - zw^{-1}}, \log \frac{1 - w^{-1}/z}{1 - W(w)^{-1}/W^*(z)} \right)
\]
of the space \( \mathcal{G}(W) \). A continuous linear functional on the space \( \mathcal{G}(W) \) is defined by taking \( (f(z), g(z)) \) into \( g(w) \) whenever \( w \) belongs to the unit disk. The reproducing kernel function for the linear functional is the element
\[
\left( \log \frac{1 - z/w^{-1}}{1 - W(z)/W^*(z)^{-1}}, \log \frac{1 - 1/[W^*(z)W(w^{-1})]}{1 - 1/(zw^{-1})} \right)
\]
of the space \( \mathcal{G}(W) \). If \( (f(z), g(z)) \) is an element of the space \( \mathcal{G}(W) \), then
\[
(\exp f(z), -z^{-1}W^*(z)e^{g(z)})
\]
is an element of the state space \( \mathcal{D}(W) \) of the canonical unitary linear system with transfer function \( W(z) \). The inequality
\[
||\exp f(z), -z^{-1}W^*(z)e^{g(z)}||_{\mathcal{D}(W)}^2 \leq \exp ||(f(z), g(z))||_{\mathcal{G}(W)}^2
\]
is satisfied. The identity
\[
\langle (\exp f(z), -z^{-1}W^*(z)e^{g(z)}), (\exp h(z), -z^{-1}W^*(z)e^{k(z)}) \rangle_{\mathcal{D}(W)} = \exp \langle (f(z), g(z)), (h(z), k(z)) \rangle_{\mathcal{G}(W)}
\]
holds for every element \( (h(z), k(z)) \) of the space \( \mathcal{G}(W) \) if equality holds.

The Grunsky Krein space \( \mathcal{G}(W) \) is defined when \( W(z) \) is a power series with constant coefficient zero which represents an injective mapping of the unit disk. A densely defined transformation \( T \) in the Dirichlet Hilbert space, which has a closed graph, takes \( f(z) \) into \( f(W(z)) \) whenever \( f(z) \) and \( f(W(z)) \) belong to the space. The spectral subspace of contractivity for the transformation is an invariant subspace for \( T^*T \), in which the restriction of \( T^*T \) is contractive, such that the orthogonal complement is an invariant subspace for the inverse of \( T^*T \), in which the restriction of the inverse of \( T^*T \) is contractive. Uniqueness of the subspace is obtained by requiring that the eigenvectors of \( T^*T \) for the eigenvalue one be included in the subspace. The Grunsky Krein space \( \mathcal{G}(W) \) is defined as the orthogonal sum of a Hilbert space and the anti-space of a Hilbert space. The Hilbert space is contained contractively in the Dirichlet Hilbert space. The adjoint of the
inclusion of the Hilbert space in the Dirichlet Hilbert space is equal to the restriction of $1 - T^*T$ to the spectral subspace of contractivity for $T$. The elements of the anti–space of a Hilbert space are of the form $f(W(z))$ with $f(z)$ in a Hilbert space which is contained contractively in the Dirichlet Hilbert space. The adjoint of the inclusion of the Hilbert space in the Dirichlet Hilbert space is the restriction of $1 - (T^*T)^{-1}$ to the orthogonal complement of the spectral subspace of contractivity for $T$. The transformation which takes $f(z)$ into $f(W(z))$ acts as an isometry on the anti–space of the Hilbert space. The elements of the space $G(W)$ are represented by power series with constant coefficient zero.

The function

$$\log \frac{1 - W(z)W(w)^{-1}}{1 - zw^{-1}}$$

of $z$ is represented by an element of the space $G(W)$ when $w$ belongs to the unit disk and acts as reproducing kernel function for function values at $w$. The Grunsky Krein space $G(W)$ is isometrically equal to the Grunski Hilbert space $G(W)$ when $W(z)$ is a power series with constant coefficient zero which represents an injective mapping of the unit disk into itself.

The Grunsky Krein space $\text{ext} G(W)$ is defined when $W(z)$ is a power series with constant coefficient zero which represents an injective mapping of the unit disk. A densely defined transformation $T$ in the Dirichlet Krein space, which has a closed graph, takes $f(z)$ into $f(W(z))$ whenever $f(z)$ and $f(W(z))$ belong to the space. The spectral subspace of contractivity for $T$ is an invariant subspace for $T^*T$, in which the restriction of $T^*T$ is contractive, such that the orthogonal complement is an invariant subspace for the inverse of $T^*T$, in which the inverse of $T^*T$ is contractive. Uniqueness of the subspace is obtained by requiring that the eigenvalue of $T^*T$ for the eigenvalue one are included in the subspace. The Grunsky Krein space $\text{ext} G(W)$ is defined as the orthogonal sum of a Hilbert space and the anti–space of a Hilbert space. The elements of the space are pairs $(f(z), g(z))$ of power series such that $f(z)$ has constant coefficient zero. An isometric transformation of the space onto a Krein space of Laurent series is defined by taking $(f(z), g(z))$ into

$$f(z) + g(z^{-1}).$$

The Krein space of Laurent series is then defined as the orthogonal sum of a Hilbert space and the anti–space of a Hilbert space. The Hilbert space is contained continuously and contractively in the Dirichlet Krein space. The adjoint of the inclusion of the Hilbert space in the Dirichlet Krein space is equal to the restriction of $1 - T^*T$ to the spectral subspace of contractivity for $T$. The elements of the anti–space of a Hilbert space are of the form $f(W(z))$ with $f(z)$ in a Hilbert space which is contained continuously and contractively in the Dirichlet Krein space. The adjoint of the inclusion of the Hilbert space in the Dirichlet Krein space is the restriction of $1 - (T^*T)^{-1}$ to the orthogonal complement of the spectral subspace of contractivity for $T$. The transformation which takes $f(z)$ into $f(W(z))$ acts as an isometry on the anti–space of the Hilbert space. A contractive partially isometric transformation of the Grunsky Krein space $\text{ext} G(W)$ onto the Grunsky Krein space $G(W^*)$ is defined by taking $(f(z), g(z))$ into

$$(g(z) - g(0), g(0) + f(z)).$$
A continuous linear functional on the Grunsky Krein space \( G(W) \) is defined by taking \((f(z), g(z))\) into \(f(w)\) when \(w\) belongs to the unit disk. The reproducing kernel function for the linear functional is

\[
\left( \log \frac{1 - W(z)W(w)^-}{1 - zw^-}, \log \frac{1 - w^-/z}{1 - W(w)^-/W(z)} \right)
\]

of the space \( G(W) \). A continuous linear functional on the space \( G(W) \) is defined by taking \((f(z), g(z))\) into \(g(w)\) when \(w\) belongs to the unit disk. The reproducing kernel function for the linear functional is the element

\[
\left( \log \frac{1 - z/w^-}{1 - W(z)/W(w)^-}, \log \frac{1 - (W^*(z)W(w^-))}{1 - 1/(zw^-)} \right)
\]

of the space \( G(W) \).

A power series \( G(z) \) is said to be subordinate to a power series \( F(z) \) if

\[
G(z) = F(W(z))
\]

for a power series \( W(z) \) with constant coefficient zero which represents a mapping of the unit disk into itself. Assume that \( F(z) \) and \( G(z) \) are power series with constant coefficient zero which represent injective mappings of the unit disk. Then \( G(z) \) is subordinate to \( F(z) \) if, and only if, the region onto which \( G(z) \) maps the unit disk is contained in the region onto which \( F(z) \) maps the unit disk. The power series \( W(z) \) then represents an injective mapping of the unit disk into itself. The identity

\[
G'(0) = F'(0)W'(0)
\]

holds with the derivative of \( W(z) \) at the origin positive and less than or equal to one if the derivatives of \( F'(z) \) and \( G(z) \) at the origin are positive. The power series \( F(z) \) and \( G(z) \) are equal if the derivative of \( W(z) \) at the origin is equal to one.

Assume that \( Z(a, z) \) and \( Z(b, z) \) are power series with constant coefficient zero which represent injective mappings of the unit disk and that the identity

\[
Z(a, z) = Z(b, W(b, a, z))
\]

holds for a power series \( W(b, a, z) \) with constant coefficient zero which represents an injective mapping of the unit disk into itself. Then the Grunsky Hilbert space \( G(W(b, a)) \) is contained continuously and contractively in the Grunsky Krein space \( G(Z(a)) \). An isometric transformation of the Grunsky Krein space \( G(Z(b)) \) onto the complementary space to the Grunsky Hilbert space \( G(W(b, a)) \) in the Grunsky Krein space \( G(Z(a)) \) is defined by taking \( f(z) \) into \( f(W(b, a, z)) \). The Grunsky Hilbert space \( \text{ext} G(W(b, a)) \) is contained continuously and contractively in the Grunsky Krein space \( \text{ext} G(Z(a)) \). An isometric transformation of the Grunsky Krein space \( G(Z(b)) \) onto the complementary space to the
Grunsky Hilbert space \( \mathcal{G}(W(b, a)) \) in the Grunsky Krein space \( \mathcal{G}(Z(a)) \) is defined by taking \((f(z), g(z))\) into

\[
(f(W(b, a, z)), g(W(b, a, z))).
\]

Subordination is a partial ordering of power series with constant coefficient zero and coefficient of \( z \) positive which represent injective mappings of the unit disk. A Löwner family is a maximal totally ordered set of such functions parametrized by derivatives at the origin. All positive numbers appear as parameters in a Löwner family. If \( F(z) \) and \( G(z) \) are power series with constant coefficient zero and coefficient of \( z \) positive which represent injective mappings of the unit disk such that \( G(z) \) is subordinate to \( F(z) \), then \( F(z) \) and \( G(z) \) are members of a Löwner family by the Zorn lemma.

A Löwner family of functions \( Z(t, z) \) satisfies the linear Löwner equation

\[
t \frac{\partial}{\partial t} Z(t, z) = \phi(t, z) z \frac{\partial}{\partial z} Z(t, z).
\]

The coefficients are a measurable family of Herglotz functions \( \phi(t, z) \) with constant coefficient one. Measurability means that the coefficient of \( z^n \) in \( \phi(t, z) \) is a Lebesgue measurable function of \( t \) for every nonnegative integer \( n \). The partial derivative with respect to \( z \) is an operation on power series. The interpretation of the equation is that the coefficient of \( z^n \) on the left is equal to the coefficient of \( z^n \) on the right for every nonnegative integer \( n \). The partial derivative with respect to \( t \) is taken in the sense of absolute continuity with respect to Lebesgue measure. Since \( Z(a, z) \) is subordinate to \( Z(b, z) \) when \( a \) is less than or equal to \( b \), the equation

\[
Z(a, z) = Z(b, W(b, a, z))
\]

admits a solution \( W(b, a, z) \) which is a power series with constant coefficient zero and coefficient of \( z \) equal to \( a/b \) which represents an injective mapping of the unit disk into itself. The coefficients of \( W(b, t, z) \) are absolutely continuous functions of in the interval \((0, b]\) which satisfy the linear Löwner equation

\[
t \frac{\partial}{\partial t} W(b, t, z) = \phi(t, z)z \frac{\partial}{\partial z} W(b, t, z).
\]

The coefficients of \( W(t, a, z) \) are absolutely continuous functions of \( t \) in the half-line \([a, \infty)\) which satisfy the nonlinear Löwner equation

\[
t \frac{\partial}{\partial t} W(t, a, z) = -\phi(t, W(t, a, z))W(t, a, z).
\]

The nonlinear Löwner equation is a consequence of the Huygens identity

\[
W(b, a, z) = W(b, t, W(t, a, z))
\]

which holds when \( t \) belongs to the interval \([a, b]\). The identity

\[
Z(a, z) = \lim bW(b, a, z)
\]
holds in the limit of large \( b \) uniformly on compact subsets of the unit disk.

Löwner families are constructed as solutions of the Löwner equation. If a measurable family of Herglotz functions \( \phi(t, z) \) with constant coefficient zero is given, a unique Löwner family of functions \( Z(t, z) \) exists which is a solution of the linear Löwner equation

\[
  t \frac{\partial}{\partial t} Z(t, z) = \phi(t, z)z \frac{\partial}{\partial z} Z(t, z)
\]

with the given coefficients. An expansion theorem for the Löwner equation is stated using measurable families of elements \( f(t, z) \) of the Herglotz spaces \( \mathcal{L}(\phi(t)) \). All positive numbers \( t \) are used as parameters. Measurability of the family means that the coefficient of \( z^n \) in \( f(t, z) \) is a Lebesgue measurable function of \( t \) for every nonnegative integer \( n \). These conditions imply that the norm of \( f(t, z) \) in the space \( \mathcal{L}(\phi(t)) \) is a measurable function of \( t \). Two such families are considered equivalent in an interval \( [a, b] \) if the elements of the spaces \( \mathcal{L}(\phi(t)) \) are equal for almost all \( t \) in the interval. A Hilbert space is formed by the equivalence classes of families \( f(t, z) \) such that the integral

\[
  \int_a^b \|f(t, z)\|_{\mathcal{L}(\phi(t))}^2 t^{-1} dt
\]

is finite. A corresponding element \( F(z) \) of the space \( \mathcal{G}(W(b, a)) \) is defined by the integral

\[
  F(z) = 2 \int_a^b f(t, W(t, a, z)) W(t, a, z) t^{-1} dt.
\]

The inequality

\[
  \|F(z)\|_{\mathcal{G}(W(b, a))}^2 \leq 2 \int_a^b \|f(t, z)\|_{\mathcal{L}(\phi(t))}^2 t^{-1} dt
\]

is satisfied. Every element \( F(z) \) of the space \( \mathcal{G}(W(b, a)) \) admits a representation for which equality holds.

A related expansion is stated using measurable families of elements \( (f(t, z), g(t, z)) \) of the extended Herglotz spaces \( \mathcal{E}(\phi(t)) \). All positive numbers \( t \) are used as parameters. Measurability means that the coefficient of \( z^n \) in \( f(t, z) \) and the coefficient of \( z^n \) in \( g(t, z) \) are Lebesgue measurable functions of \( t \) for every nonnegative integer \( n \). Two such families are considered equivalent in an interval \( [a, b] \) if the elements of the spaces \( \mathcal{E}(\phi(t)) \) are equal for almost all \( t \) in the interval. A Hilbert space is formed by the set of equivalence classes of families of pairs \( (f(t, z), g(t, z)) \) such that the integral

\[
  \int_a^b \|(f(t, z), g(t, z))\|_{\mathcal{E}(\phi(t))}^2 t^{-1} dt
\]

is finite. For every such family an element \( (F(z), G(z)) \) of the space \( \text{ext} \mathcal{G}(W(b, a)) \) is defined by the integrals

\[
  F(z) = 2 \int_a^b f(t, W(t, a, z)) W(t, a, z) t^{-1} dt
\]

\[
  G(z) = 2 \int_a^b g(t, W(t, a, z)) W(t, a, z) t^{-1} dt
\]
and

\[ G(z) = 2 \int_a^b g(t, W(t, a, z)) t^{-1} dt. \]

The inequality

\[ \| (F(z), G(z)) \|_{\text{ext } G(W(b, a))}^2 \leq 2 \int_a^b \| (f(t, z), g(t, z)) \|_{\mathcal{E}(\phi(t))}^2 t^{-1} dt \]

is satisfied. Every element \((F(z), G(z))\) of the space \(\text{ext } G(W(b, a))\) admits a representation for which equality holds.