Compositional roots of Hénon maps

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Abstract

Let $H$ denote a composition of complex Hénon maps in $\mathbb{C}^2$. In this paper we show that the only possible compositional roots of $H$ are also compositions of Hénon maps, and that $H$ can have compositional roots of only finitely many distinct orders.

1 Introduction

Following [BS], we say that a generalized Hénon map is a map of the form

$$H(z, w) = (w, p(w) - az),$$

where $p$ is a monic polynomial of degree $d \geq 2$ and $a \in \mathbb{C} - \{0\}$, and we let $\mathcal{G}$ denote the space of finite compositions of such maps. From [FM], we know that any polynomial diffeomorphism of $\mathbb{C}^2$ is conjugate either to one of the maps in $\mathcal{G}$ or to an elementary map which preserves each line of the form $w = \text{const}$. In [BF], we classified, up to conjugacy, all polynomial diffeomorphisms which arise as the time-1 map of a holomorphic vector field. In particular, each of these maps is an elementary map and has compositional roots of all orders. Moreover, in some cases, these roots can be nonpolynomial. See [AF] for information about such cases.

In this paper we treat the question of the existence of compositional roots for the remaining cases. In particular, we show that any root of a map in $\mathcal{G}$ must be a polynomial map and that any map in $\mathcal{G}$ can have roots of only a finite number of distinct orders. For the remaining elementary maps which are not the time-1 map of a flow, we show that such maps have roots of arbitrarily high order and nonpolynomial roots, but that any root of such a map is conjugate to a polynomial elementary map.

2 Dynamical behavior and Green’s functions

Fix $H \in \mathcal{G}$. Let $K^+(= K^+(H))$ and $K^- (= K^-(H))$ denote the set of points $p$ in $\mathbb{C}^2$ such that the orbit of $p$ under $H$ is bounded with respect to forward or backward iteration, respectively.

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Also, for \( R > 0 \), define sets
\[
V^- := \{(z, w) : |w| > R \text{ and } |w| > |z|\},
\]
\[
V^+ := \{(z, w) : |z| > R \text{ and } |w| < |z|\},
\]
\[
V := \{(z, w) : |z| \leq R \text{ and } |w| \leq R\}.
\]

A simple argument shows that \( K^\pm \subseteq V^\pm \cup \overline{V} \) for \( R \) sufficiently large. Finally, we let \( d \) denote the degree of \( H \) as a polynomial map, and define functions
\[
G^\pm(z, w) := \lim_{n \to \infty} \frac{1}{d^n} \log^+ \|H^\pm_n(z, w)\|.
\]

It is immediate that \( G^+ \) is continuous and plurisubharmonic (psh) on \( \mathbb{C}^2 \), identically 0 on \( K^+ \), and strictly positive and pluriharmonic on \( \mathbb{C}^2 - K^+ \). Analogous statements are true with \( G^- \) and \( K^- \) in place of \( G^+ \) and \( K^+ \). See, for example, [BS] for a more systematic discussion.

Note also that \( G^\pm \circ H = d^{\pm 1} G^\pm \). Moreover, corollary 2.6 in [BS] implies the following.

**Lemma 2.1** There exist \( R > 0 \), \( C > 0 \) such that if \((z, w) \in \overline{V^-} \cup \overline{V}\), then
\[
\log^+ |w| - C \leq G^+(z, w) \leq \log^+ |w| + C,
\]
and if \((z, w) \in \overline{V^+} \cup \overline{V}\), then
\[
\log^+ |z| - C \leq G^-(z, w) \leq \log^+ |z| + C.
\]

Since \( G^+(z_0, w) \) is subharmonic in \( w \) for each fixed \( z_0 \), we see that it is bounded from above in \( \{|w| < |z_0|\} \) by its maximum on the boundary of this disk, which is contained in \( \overline{V^-} \cup \overline{V} \). Applying a similar argument to \( G^- \) gives the following.

**Lemma 2.2** There exists \( C > 0 \) such that for \((z, w) \in \mathbb{C}^2\),
\[
G^\pm(z, w) \leq \max\{|\log^+ |z|, \log^+ |w|\} + C.
\]

### 3 Compositional roots and Green’s functions

Fix \( H \in \mathcal{G} \) and let \( d \) be the degree of \( H \). In this section we show that if \( F^n = H \) in the sense of composition, then \( G^\pm \circ F = d^{\pm 1/n} G^\pm \). First a simple lemma.

**Lemma 3.1** Suppose \( F \) is an automorphism of \( \mathbb{C}^2 \) and \( F^n = H \). Then \( K^+ \) and \( K^- \) are the same for \( F \) as for \( H \), and \( F \) is a diffeomorphism of \( K^+ \) and of \( K^- \).

**Proof:** Take \( p \in K^+(H) \) and let \( O^+_H(p) \) denote the forward orbit of \( p \) under \( H \). Then
\[
\bigcup_{j=0}^{n-1} F^j(O^+_H(p)) \text{ is compact, and } O^+_F(p) \text{ is contained in this set, hence is bounded. Thus } K^+(H) \subseteq K^+(F).
\]
If \( p \notin K^+(H) \), then the forward orbit is not bounded for \( H = F^n \), hence is not bounded for \( F \). Thus \( K^+(H) = K^+(F) \), and a similar argument applies to \( K^- \).

The fact that \( F \) is a diffeomorphism of \( K^\pm = K^\pm(F) \) is clear from the definition of these sets. ■
LEMMA 3.2 Let $F$ be as in the previous lemma. Then $G^+ \circ F = d^{\pm 1/n} G^\pm$.

Proof: Since $F$ is holomorphic on $\mathbb{C}^2$ and preserves $K^+$, we see that $G^+ \circ F$ is 0 on $K^+$, plurisubharmonic and continuous on $\mathbb{C}^2$, and strictly positive and pluriharmonic on $\mathbb{C}^2 - K^+$.

Fix $z_0$ and define $g_{z_0}(w) := G^+ \circ F(z_0, w)$. Note that $K^+ \cap (\{z_0\} \times \mathbb{C})$ is a compact set and that $g$ is harmonic on the complement of this set. Hence, outside a large disk, $g_{z_0}$ has a harmonic conjugate in a neighborhood of any point. Using analytic continuation in the exterior of this disk, we obtain a harmonic conjugate with periods. Hence for some $r > 0$, some constant $c_{z_0}$, and a real harmonic function $h_{z_0}$, we get a function

$$
\phi_{z_0}(w) = g_{z_0}(w) - c_{z_0} \log |w| + ih_{z_0}(w)
$$

which is holomorphic for $|w| > r$.

Since $g_{z_0} \geq 0$, we have $|\exp(-\phi_{z_0}(w))| \leq |w|^{c_{z_0}}$. Hence $\exp(-\phi_{z_0}(w))$ has at most a pole at infinity, so we can write

$$
\exp(-\phi_{z_0}(w)) = w^N \exp(f(w))
$$

for some integer $N$ and some $f$ holomorphic in $|w| > r$ with a removable singularity at infinity.

Taking absolute value and log, we get $g_{z_0}(w) - c_{z_0} \log |w| = -N \log |w| - \text{Re}(f(w))$. Hence $g_{z_0}(w) = b_{z_0} \log |w| + O(1)$ in $\{|w| > 2r\}$, for some $b_{z_0}$. Since $g_{z_0} \geq 0$, we have $g_{z_0}(w) = b_{z_0} \log^+ |w| + O(1)$ in $\mathbb{C}$.

We claim that $b_z$ is independent of $z$. Note that $2\pi b_{z_0}$ is the period for the harmonic conjugate of $g_{z_0}$ in $|w| > r$, and that $g_z(w)$ is pluriharmonic in $(z, w)$ near $(z_0, w_0)$ for any $|w_0| > r$.

Fix $|w_0| > r$. We can construct the harmonic conjugate for $g$ in the bidisk $\Delta(z_0; r_0) \times \Delta(w_0; r_0)$ for some $r_0$ small. For $w \in \Delta(w_0; r_0)$, we can use analytic continuation as above to extend $g_z$ around a circle in $\{z\} \times \{|w| > r\}$. Doing this for each such $w$ gives a new harmonic conjugate for $g$ in the bidisk, which must differ from the original by a constant. Thus $b_z = b_{z_0}$ for $z$ near $z_0$.

Hence $g_{z_0}(w) = b \log^+ |w| + O(1)$, and from lemma 2.1, we see $G^+(z_0, w) = \log^+ |w| + O(1)$. Thus $g_{z_0}(w) - bG^+(z_0, w)$ is continuous for $w \in \mathbb{C}$ and harmonic for $w$ such that $(z_0, w) \notin K^+$, has a removable singularity at $\infty$, and is 0 for $w$ such that $(z_0, w) \in K^+$, which is a nonempty set. Hence $g_{z_0}(w) \equiv bG^+(z_0, w)$ for all $w \in \mathbb{C}$.

Similarly, $G^+ \circ F(z, w) - bG^+(z, w) \equiv 0$ for all $(z, w) \in \Delta(z_0; r_0) \times \mathbb{C}$. Since this difference is pluriharmonic in $\mathbb{C}^2 - K^+$, which is connected, it must be 0 throughout $\mathbb{C}^2 - K^+$, hence throughout $\mathbb{C}^2$ since both terms are 0 on $K^+$.

Finally, $G^+ \circ F^n(z, w) = G^+ \circ H(z, w) = dG^+(z, w)$, while induction with the above result shows that $G^+ \circ F^n(z, w) = b^n G^+(z, w)$ Hence $b^n = d$, and $b > 0$ since $G^+ \geq 0$. This gives the lemma for $G^+$, and the same proof applies to $G^-$.
4 Polynomial roots

In the proof of the following theorem, we use the terminology and results of [FM]. In particular, we use the fact that the group of polynomial automorphisms of $\mathbb{C}^2$ is the amalgamated product of the group $A$ of affine linear automorphisms and the group $E$ of elementary automorphisms which preserve the set of lines of the form $w = \text{const}$. A reduced word is an automorphism of the form $g_1 \cdots g_k$, $k \geq 1$, where each $g_k$ is in $A$ or $E$ but not in the intersection of these two groups and no two adjacent $g_j$’s are in the same group. We say that $k$ is the length of this reduced word. Also, we need to know that the identity cannot be written as a reduced word.

**THEOREM 4.1** Suppose $H \in \mathcal{G}$ is a composition of generalized Hénon maps and $F$ is an automorphism of $\mathbb{C}^2$ with $F^n = H$. Then $F \in \mathcal{G}$.

**Proof:** Let $(z, w) \in \mathbb{C}^2$, and let $F = (F_1, F_2)$. If $F(z, w) \in V^- \cup V$, then from lemmas 2.1, 3.2, and 2.2, we see

$$\log^+ |F_2| - C_1 \leq G^+(F(z, w)) = d^{1/n} G^+(z, w) \leq d^{1/n} (\max\{\log^+ |z|, \log^+ |w|\} + C_2).$$

Exponentiating and using $|F_1| \leq |F_2| + R$, we obtain

$$|F(z, w)| \leq C \max\{(|z| + 1)^{d^{1/n}}, (|w| + 1)^{d^{1/n}}\}.$$  

Similarly, if $F(z, w) \in V^+$, then

$$\log^+ |F_1| - C_1 \leq G^-(F(z, w)) = 1/d^{(1/n)} G^-(z, w) \leq 1/d^{(1/n)} (\max\{\log^+ |z|, \log^+ |w|\} + C_2).$$

Exponentiating and using $|F_2| \leq |F_1|$, we obtain

$$|F(z, w)| \leq C \max\{(|z| + 1)^{1/d^{(1/n)}}, (|w| + 1)^{1/d^{(1/n)}}\}.$$  

Hence $F$ has polynomial growth throughout $\mathbb{C}^2$, hence must be a polynomial.

We show next that $F \in \mathcal{G}$. Let $\tau(z, w) := (w, z)$. By [FM], we can write $H = \tau e_1 \cdots \tau e_m$, for some elementary maps $e_j$. Since $F^n = H$, $F$ must be a reduced word with length at least 2. There are four possibilities for the form of $F$. The first is

$$F = a_1 e'_1 \cdots a_t e'_t$$

for some affine, non-elementary maps $a_j$ and some elementary, non-affine maps $e'_j$. By [FM] or [AR], each $a_j$ can be written $a_j = a_j^1 \tau a_j^2$, where $a_j^1$ and $a_j^2$ are affine and elementary, and $a_1^1$ has the form $a_1^1(z, w) = (bz + cw, w)$. 

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Now, since $a_k^j$ is elementary, $F^n$ has the form $a_1^1 \circ (p, q)$, where $p$ and $q$ are polynomials and $(p, q) = \tau e''_1 \cdots \tau e''_{nl}$. By [FM], we have $\deg(q) > \deg(p)$, and likewise the degree of the second coordinate function of $H = F^n$ is larger than the degree of the first coordinate function. This implies that $c = 0$, so $a_1^1(z, w) = (bz, w)$. Replacing $a_2^1$ by $\sigma a_2^1$, where $\sigma(z, w) = (z, bw)$, we obtain
\[ F = \tau E_1 \cdots \tau E_l. \]
Hence $F \in \mathcal{G}$.

The second case is
\[ F = e'_1 a_2 \cdots a_l e'_l. \]
In this case, we can replace each $a_j$ by $a_1^1 \tau a_2^2$ as before, and hence relabeling, we may assume $F = e'_1 \tau \cdots \tau e'_l$. But then $H = F^n = e'_1 \tau \cdots \tau e'_{nl}$, which implies that the degree of the first coordinate of $H$ is larger than the degree of the second coordinate, which is impossible. Hence $F$ cannot have this form.

The third case is
\[ F = e'_1 a_2 \cdots e'_{l-1} a_l. \]
As before, we may relabel to assume that $F = e'_1 \tau \cdots e'_{l-1} \tau a_2^2$. But then $H = F^n = e'_1 \tau \cdots e'_k \tau a_2^2$, but also $H = \tau e_1 \cdots \tau e_m$. Hence
\[ I = (F^n)^{-1} H = (a_2^2)^{-1} (\tau e'_k)^{-1} \cdots (e'_1)^{-1} (\tau e_1 \cdots \tau e_m). \]
But then $I$ has been written as a reduced word, which is impossible from [FM]. Thus $F$ cannot have this form.

In the final case, we have
\[ F = a_1 e'_1 \cdots e'_{l-1} a_l. \]
Again we may relabel and collect terms and assume $H = F^n = a_1^1 \tau e'_1 \cdots e'_{k-1} \tau a_2^2$. Since $a_2^2$ is linear, we can use an argument like that in the first case to relabel and replace $a_1^1$ by the identity. Since $a_1^1$ is elementary, we can write $a_1^1(z, w) = (az + bw + c, dw + e)$ with $a \neq 0$. Applying $\tau e'_1 \cdots e'_{k-1} \tau$ to this, we see that the homogeneous polynomial of highest degree in $F^n$ depends on $z$. But a simple inductive argument shows that the corresponding polynomial for $H$ is independent of $z$. Hence $F$ cannot have this form.

Thus $F \in \mathcal{G}$. $\blacksquare$

Remark 4.2 In general, a map $H$ can have distinct roots of a given order. For example, the map $H$ given by squaring $F(z, w) = (w, z + w^2)$ has three square roots. This is true because $(F \circ s)^2 = H$ for $s(z, w) = (\omega^2 z, \omega w)$, where $\omega^3 = 1$. In fact, one can check that these are the only possible square roots of $H$.

5 Roots of elementary maps

In [BF], we showed that no Hénon map can be the time-1 map of the flow of a holomorphic vector field and gave a precise classification of those maps which can be the time-1 map of
such a flow. In [AF] and [AFV], it was shown that any flow of a holomorphic vector field whose time-1 map is an elementary map is in fact conjugate to a flow which is polynomial for all time.

In this section, we consider the set of elementary maps which are not the time-1 map of any holomorphic flow and show that such maps have roots of arbitrarily high order but that any root is conjugate to a polynomial map.

The elementary maps which cannot be the time-1 map of a flow have the form

$$F(z, w) = (\beta^\mu(z + w^\mu q(w)), \beta w),$$

where $\beta$ is a primitive $r$th root of unity, $q(w) = w^k + q_{k-1}w^k + \cdots + q_1w + q_0$, and $k \geq 1$. A simple check shows that if we replace $w^\mu q(w^r)$ by $(w^\mu q(w^r))/(l+1)$ for any $l \in \mathbb{Z}^+$, then the resulting map is an $(lr+1)$st root of $F$.

In general, maps of this form can have nonpolynomial roots. For instance, let $F(z, w) = (-z + w(w^4 + 1), -w)$ and let $k$ be any entire function of one variable. Define $\phi(z, w) = (iz + w(w^4+1)/2+w^3k(w^4)), iw)$. A simple check shows that $\phi^2 = F$, and $\phi$ is nonpolynomial whenever $k$ is transcendental.

We claim that any root of $F$ is conjugate to a polynomial automorphism. Suppose that $\phi$ is an automorphism of $\mathbb{C}^2$ with $\phi^a = F$. Then $\phi F^r \phi^{-1} = F^r$, so an argument like that in [FM, theorem 6.10] shows that $\phi(z, w) = (e^{g(w)}z + h(w), aw + b)$ for some entire $g, h$, and some $a, b \in \mathbb{C}$, $a \neq 0$. Since $\phi^a = F$, we see that $a^n = \beta$ and $b(a^n - 1)/(a - 1) = 0$, so that $b = 0$.

Using this form for $\phi$ and the fact that $\phi F^r = F^r \phi$, it follows that $e^{g(w)}$ is a nonzero rational function, hence is a constant, $c \neq 0$. Moreover, since $\phi F = F \phi$, we see that $c = a^\mu$. Thus $\phi(z, w) = (a^{\mu}z + h(w), aw)$.

Now, since $\phi^a = F$, it follows that $\sum_{j=0}^{a^n-1}(a^\mu)^{-j} h(a^jw) = rw^\mu q(w^r)$. Write $h = h_1 + h_2$, where $h_2 = w^{a+kr}h_2$ for some entire $h_2$. Then the sum just given is valid with $h_2$ in place of $h$ and 0 in place of $rw^\mu q(w^r)$. Hence by [AF], there exists $f$ entire such that $f(aw) - a^n f(w) = h_2(w)$.

A simple check shows that if $\psi(z, w) = (z + f(w), w)$, then $\psi^{-1} \phi \psi$ is a polynomial, and in fact, $\psi^{-1} \phi^n \psi = F$. Thus any root of $F$ is conjugate to a polynomial automorphism.

Given any elementary map $F$ and a root $\phi^a = F$, one can ask if $\phi$ is conjugate to a polynomial automorphism. There are a few cases such as the above where this result is relatively straightforward, but in general, this seems to be a hard question. For some results along these lines in the case $F = I$, see [AR].

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