Nondensity of stability for polynomial automorphisms of $\mathbb{C}^2$

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Abstract

In the space of polynomial automorphisms of $\mathbb{C}^2$, the set of structurally stable maps is not dense. To obtain this result we develop some of the theory of moduli of stability for holomorphic maps. Also, the set of hyperbolic maps is not dense in the set of polynomial automorphisms; however, given any polynomial automorphism, F, of $\mathbb{C}^2$, there is a polynomial automorphism, G, of the same degree as and arbitrarily near F such that each periodic point of G is hyperbolic.

1 Introduction

In this paper we consider questions of dynamical stability; i.e., when do small changes in a map lead to small changes in the dynamics? More precisely, let $\mathcal{G}$ be a topological space of diffeomorphisms of a manifold M. A map $F \in \mathcal{G}$ is *structurally stable* if there exists a neighborhood $\mathcal{U}$ of F such that for each $G \in \mathcal{U}$, F and G are conjugate. In the study of dynamics of one complex variable, the question of the density of stability was resolved in [MSS] and [MS], where it was shown that in a family of rational maps of the Riemann sphere which depends holomorphically on a parameter, the set of structurally stable maps is dense. In contrast, for general C^k diffeomorphisms of a compact surface, the set of structurally stable maps is not dense in most families; e.g., [I, Ch. 7, Sec. IV].

In this paper, we show that the situation for polynomial automorphisms of $\mathbb{C}^2$ of fixed degree is analogous to that for C^k diffeomorphisms of a compact surface: structural stability is not dense. In doing so, we develop some of the theory of moduli of stability along the lines of [NPT]. In this particular case, there are numerical invariants associated with homoclinic and heteroclinic tangencies which can prevent conjugation between maps having different values for these invariants. We also show that hyperbolic maps are not dense in the space of polynomial automorphisms of sufficiently high degree, but that given a polynomial automorphism of degree d having nontrivial dynamics, there is a nearby automorphism of the same degree such that all of its periodic points are hyperbolic. This latter result is one half of the Kupka-Smale theorem in the setting of polynomial automorphisms.

Friedland and Milnor initiated the study of the dynamics of polynomial automorphisms of $\mathbb{C}^2$ in [FM]. There they distinguish between *elementary* automorphisms, which are polynomially conjugate to an automorphism of the form $(x, y) \mapsto (ax + p(y), cy + d)$ (p polynomial, $a, c \neq 0$) and which have simple dynamics, and the remaining automorphisms, which are termed *nonelementary* and which do not have simple dynamics.

Bedford and Smillie [BS2] introduced the notion of the *dynamical degree* of a polynomial automorphism of $\mathbb{C}^2$. Letting $\deg F$ denote the maximum of the degrees of the polynomial map F, the dynamical degree is defined as

$$d = d(F) = \lim_{n\to\infty} (\deg F^n)^{1/n},$$

where F^n denotes the nth iterate of F. This degree is a conjugacy invariant, and any nonelementary polynomial automorphism of $\mathbb{C}^2$ is conjugate to an automorphism whose dynamical degree is equal to its degree as a polynomial. Moreover, the elementary polynomial automorphisms are exactly those which have dynamical degree 1.

DEFINITION 1.1 *Let $\mathcal{P}_d$ denote the set of polynomial automorphisms of dynamical degree d.*

The topology on $\mathcal{P}_d$ is that induced by the compact-open topology applied both to an automorphism and its inverse. With this topology, $\mathcal{P}_d$ is complete. This topology can also be induced by a complete metric.

The main theorem of this paper is the following.

THEOREM 1.2 *There exists $N \in \mathbb{Z}^+$ sufficiently large such that if $d \geq N$, then the set of structurally stable maps in $\mathcal{P}_d$ is not dense in $\mathcal{P}_d$.*

In proving the previous theorem, we develop some techniques which lead to the following result.

THEOREM 1.3 *Let $F \in \mathcal{P}_d$, $d \geq 1$, and let $\mathcal{U}$ be a neighborhood of F. Then there exists $G \in \mathcal{U}$ such that each periodic point of G is hyperbolic.*

As noted earlier, this result is one half of the Kupka-Smale theorem for polynomial automorphisms. It is an open question if the part of the Kupka-Smale theorem giving the transversality of stable and unstable manifolds is also true in this setting. The full Kupka-Smale theorem holds in the space of all holomorphic automorphisms of $\mathbb{C}^n$ [B2], but the techniques used there to prove transversality involve composing a given automorphism with other automorphisms constructed to move the stable manifold in a specified manner. Doing this within $\mathcal{P}_d$ requires that all the additional automorphisms are linear, which is not sufficient for the argument in [B2].

In the next section we show that the order of contact between two 1-dimensional complex manifolds in $\mathbb{C}^2$ is a topological invariant, then construct a family of automorphisms whose fixed points have associated eigenvalues which are nonconstant functions of the parameter. In the final section we adapt the methods of [NPT] to the complex case to show that some differential information is preserved even under topological conjugacy, thus giving rise to moduli of stability. Combining these results with results of [B1] gives theorem 1.2.

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2 Preliminary results

DEFINITION 2.1 *Let F be a smooth map of a manifold M. A point $p \in M$ is said to be nonwandering for F if for each neighborhood V of p, there exists $n > 0$ such that $F^n(V) \cap V \neq \emptyset$. The nonwandering set of F, $\Omega = \Omega(F)$ is the set of all nonwandering points of F.*

Recall that a periodic point p of period n of a diffeomorphism F is *hyperbolic* if none of the eigenvalues of D_pF^n has modulus 1. Moreover, a hyperbolic periodic point has stable and unstable manifolds, and these manifolds are immersed 1-dimensional complex manifolds when F is holomorphic. Recall also that $F \in \mathcal{P}_d$ is *hyperbolic* if there is a continuous, F-invariant splitting $E^s \oplus E^u = T\mathbb{C}^n|\Omega$ and constants $C > 0$, $0 < \mu < 1$ such that $\|DF^n|E^s\| \leq C\mu^n$ and $\|DF^{-n}|E^u\| \leq C\mu^n|$ for $n \geq 0$. See [R] for further background in the real case.

The following theorem on the nondensity of hyperbolicity is an analytic counterpart of theorem 1.2, which is topological in nature. It should be noted that the density of hyperbolicity for polynomials of one variable is an open question for every degree $d \geq 2$.

THEOREM 2.2 *There exists $N \in \mathbb{Z}^+$ such that if $d \geq N$, then the set of hyperbolic maps is not dense in $\mathcal{P}_d$.*

Proof: From [B1], there exists $N \in \mathbb{Z}^+$ such that if $d \geq N$, then there is an open set $\mathcal{U} \subseteq \mathcal{P}_d$ such that each $F \in \mathcal{U}$ has a tangency between the stable and unstable manifolds for a basic set, Λ. Such a point of tangency, p, is in $\Omega(F)$, but it is immediate that such a point prevents a hyperbolic splitting of $\Omega(F)$. Since this is true for all maps in U, we see that hyperbolicity is not dense in $\mathcal{P}_d$ for d large. ∎

The next lemma implies that the order of tangency between two 1-dimensional complex manifolds in $\mathbb{C}^2$ is a topological invariant.

LEMMA 2.3 *Suppose M_1 and M_2 are two 1-dimensional complex submanifolds of $\mathbb{C}^2$ which intersect in exactly one point p. Let U be a neighborhood of p, and suppose that $\phi : U \to \mathbb{C}^2$ is continuous and injective and that $\phi(M_j)$ is also a complex manifold, $j = 1, 2$. Then the order of contact between $\phi(M_1)$ and $\phi(M_2)$ is the same as that between M_1 and M_2.*

Proof: Using a topological change of coordinates, we may assume that p and $\phi(p)$ are both the origin, that M_1 and $\phi(M_1)$ are contained in $\mathbb{C} \times \{0\}$, and that M_2 and $\phi(M_2)$ are contained in the graphs of $z \mapsto z^m$ and $z \mapsto z^n$, respectively, for some positive integers m and n.

Let Δ be the unit disk in $\mathbb{C}$ and $\Delta^* = \Delta \setminus \{0\}$. Consider an element in the fundamental group $\pi_1(\Delta \times \Delta^*)$ of the form $\gamma(\theta) = (\delta e^{i\theta}, \delta e^{im\theta})$, for small $\delta > 0$. Then $[\gamma] = [m] \in \pi_1(\Delta \times \Delta^*)$, and since ϕ is a homeomorphism of a neighborhood of the origin which preserves the z-axis, $[\phi \circ \gamma] = [m] \in \pi_1(\Delta \times \Delta^*)$. From the topological change of coordinates introduced above, the image of $\phi \circ \gamma$ is contained in the graph of $z \mapsto z^n$ minus the origin. In particular, the projection of $\phi \circ \gamma$ to the z-axis is an element of $\pi_1(\Delta^*)$, and taking the graph of this projection under $z \mapsto z^n$ shows that $\phi \circ \gamma \in n\mathbb{Z}$ in $\pi_1(\Delta \times \Delta^*)$. Thus $m \in n\mathbb{Z}$, and by a symmetric argument we have $n \in m\mathbb{Z}$, so $n = m$ as desired. ∎

In the following proposition, JF represents the determinant of the Jacobian matrix of partial derivatives of F. Note that JF is constant for a polynomial automorphism F.

PROPOSITION 2.4 *Let $F \in \mathcal{P}_d$ with $d \geq 2$. Then there exists a one parameter family $\{F_\mu\}_{\mu \in \mathbb{C}^*} \subseteq \mathcal{P}_d$ with $F_1 = F$ such that for all $n \geq 1$ there exists a discrete subset $E_n \subseteq \mathbb{C}^*$ such that if $\mu \in \mathbb{C}^* \setminus E_n$, then F_μ^n has exactly d^n distinct fixed points, each of which varies holomorphically with μ. For each such fixed point $p(\mu)$, the eigenvalues of $D_{p(\mu)}F^n$ are distinct and are nonconstant holomorphic functions of μ. Also, JF_μ is independent of μ.*

Proof: Using [FM], we may assume that F is a composition of generalized Hénon maps of the form $H_j(z,w) = (w, p_j(w) - a_j z)$, where p_j is a monic polynomial of degree $d_j \geq 2$ and $a_j \neq 0$.

Suppose that

$$D_{(0,0)}F = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}. \tag{2.1}$$

For $p \in \mathbb{C}^2$, let $\psi_p(z,w) = (z,w) + p$. Replacing F by $\psi_p^{-1}F\psi_p$ for some p near 0, we may assume that $\delta \neq 0$. We construct F_μ with this new F, then the final family is given by $\psi_p F \psi_p^{-1}$.

For $\mu \in \mathbb{C}^*$, let $\phi_\mu(z,w) = (\mu z, w/\mu)$, and let $F_\mu = \phi_\mu F \phi_\mu$. Since ϕ_μ is volume preserving, the Jacobian of F_μ is independent of μ.

From [BS1], there exists $R > 0$ such that for each j and each $r \geq R$, the set $V_r^+ = \{(z,w) : |w| \geq r, |z| \leq |w|\}$ satisfies $H_j(V_r^+) \subseteq V_{2r}^+$, and likewise $V_r^- = \{(z,w) : |z| \geq r, |w| \leq |z|\}$ satisfies $H_j^{-1}(V_r^-) \subseteq V_{2r}^-$. In particular, this implies that the nonwandering set of F (hence the set of periodic points) is contained in $\Delta^2(0;R)$. A simple check shows that for $|\mu| < 1$, $F_\mu(V_{|\mu|R}^+) \subseteq V_{2|\mu|R}^+$ and $F_\mu^{-1}(V_{|\mu|R}^-) \subseteq V_{2|\mu|R}^-$, and hence all periodic points for F_μ are contained in $\Delta^2(0;|\mu|R)$.

Let λ_1, λ_2 be the eigenvalues of $D_{(0,0)}F_\mu$ with $|\lambda_1| \leq |\lambda_2|$. Calculating λ_1 and λ_2 and using $\alpha\delta - \beta\gamma \neq 0$, we see that $\lim_{\mu\to 0} \lambda_1 = 0$ and $\lim_{\mu \to 0} \lambda_2 = \infty$, since $\delta \neq 0$.

Hence for μ sufficiently near 0, the eigenvalues of $D_{(0,0)}F_\mu$ are distinct and have modulus different from 1. Moreover, since each fixed point, $p(\mu)$, of F_μ^n is contained in $\Delta^2(0;|\mu|R)$, the derivative $D_{p(\mu)}F_\mu^n$ will be nearly $(D_{(0,0)}F_\mu)^n$ for μ near 0. Since F_μ^n has at most d^n fixed points by [FM], we can choose μ sufficiently near 0 that all of the fixed points of F_μ^n are hyperbolic with distinct eigenvalues.

From [FM], F_μ^n has exactly d^n fixed points counted with multiplicity, and since a hyperbolic fixed point has multiplicity 1 by the inverse function theorem, the argument just given implies that F_μ^n has d^n distinct fixed points for μ sufficiently near 0. In fact, for all μ outside a discrete subset of $\mathbb{C}$, F_μ^n has d^n distinct fixed points. To see this, note that the set $A = \{(z,w,\mu) : F_\mu^n(z,w) = (z,w)\}$ is a 1-dimensional analytic set in $\mathbb{C}^2 \times \mathbb{C}^*$ and that the projection of A to $\mathbb{C}^*$ is proper with finite fibers. Since the cardinality of the fibers is constant except possibly on a 0-dimensional analytic subset where one of the eigenvalues of a fixed point of F_μ^n is 1, we see that for μ outside a discrete set, F_μ^n has d^n distinct fixed points. For such μ, the implicit function theorem implies that each fixed point varies holomorphically with μ.

The sum of the eigenvalues of DF_μ^n is a holomorphic function on $\mathbb{C}^2 \times \mathbb{C}^*$, hence on A. Since the product of the eigenvalues is JF_μ^n, which is constant, it follows that each eigenvalue is a holomorphic function of μ whenever the two eigenvalues are distinct and A is unbranched. Removing a zero-dimensional set, E_n, of μ's at which any fixed point has an associated eigenvalue equal to 1 or any eigenvalue is repeated, we are left with an open dense set of μ's which is the complement of a discrete set and on which each eigenvalue of each fixed point of F_μ^n is a holomorphic function of μ. In view of the calculation of $\lim_{\mu\to 0} \lambda_j$ given above, each eigenvalue is nonconstant on this set. ∎

With the previous result, we can prove theorem 1.3 as a corollary.

Proof of theorem 1.3: Composing with a linear contraction near the identity, we may assume that $|JF| \neq 1$. Moreover, if F has dynamical degree 1, then it is elementary in the sense of [FM], and a simple calculation shows that the the proper choice of the linear contraction makes all periodic points hyperbolic. Hence we may assume that $d \geq 2$.

With this modified F, let F_μ be as in the previous proposition. From that result, for each $n \geq 1$, there exists an open dense set $U_n = C^* \setminus E_n$ such that for each $\mu \in U_n$, F_μ^n has d^n distinct fixed points, and the corresponding eigenvalues vary holomorphically with μ and are nonconstant functions of μ. For each n, let U_n' denote the subset of U_n such that for each $\mu \in U_n'$, there exists a fixed point $p(\mu)$ of F_μ^n such that some eigenvalue of $D_{p(\mu)}F_\mu^n$ has modulus 1. Since U_n' is a real-analytic subset of U_n of codimension 1, we see that $U_n \setminus U_n'$ is open and dense in U_n, hence in $\mathbb{C}^*$. Taking the intersection of $U_n \setminus U_n'$ over all $n \geq 1$ gives a dense $\mathcal{G}_\delta$ subset V of $\mathbb{C}^*$, so in particular there exists $\mu \in V$ near enough to 1 that F_μ in $\mathcal{U}$. Taking $G = F_\mu$ gives the theorem. ∎

Remark: Note that proposition 2.4 gives a way to construct a hyperbolic automorphism using the family F_μ. For μ near 0, the nonwandering set for F_μ will be contained in $\Delta^2(0; |\mu|R)$. On this bidisk, DF_μ will map a cone field in the vertical direction into itself with a rate of expansion bounded above one, and likewise DF_μ^{-1} will map a cone field in the horizontal direction into itself with a rate of expansion bounded above 1. This implies that F_μ is hyperbolic on its nonwandering set.

3 Moduli of stability and proof of main theorem

The next lemma establishes the existence of moduli of stability for holomorphic maps: for a homoclinic tangency, the ratio of the logs of the absolute values of the eigenvalues of the associated fixed point is preserved under topological conjugacy. The methods used here are similar to those found in [NPT]. As in that work, we state the results here in terms of a heteroclinic tangency.

For this lemma, let F_1, F_2 be holomorphic automorphisms of $\mathbb{C}^2$, let p_j and q_j be fixed saddle points for F_j, and suppose that there is a tangency between $W^u(p_j, F_j)$ and $W^s(q_j, F_j)$ at the point r_j. Let U_j be a neighborhood of the orbit of r_j under F_j with $p_j, q_j \in U_j$, and suppose that $\phi : U_1 \to U_2$ is a homeomorphism satisfying $\phi(p_1) = p_2$, $\phi(q_1) = q_2$, $\phi(r_1) = r_2$, and $\phi \circ F_1 = F_2 \circ \phi$ on $U_1 \cap F_1^{-1}(U_1)$, so that ϕ gives a local conjugacy between F_1 and

F_2. In particular, since the stable and unstable manifolds are topological objects, ϕ maps $W^s(p_1, F_1) \cap U_1$ to $W^s(p_2, F_2) \cap U_2$, and likewise for the unstable manifolds.

LEMMA 3.1 *Let α_j be the contracting eigenvalue of $D_{p_j}F_j$ and let β_j be the expanding eigenvalue of $D_{q_j}F_j$. Then*

$$\frac{\log|\alpha_1|}{\log|\beta_1|} = \frac{\log|\alpha_2|}{\log|\beta_2|}.$$

Proof: Note that the stable and unstable manifolds have only a finite order of contact since otherwise they would agree on an open set, hence everywhere, which would imply that their union is a compact complex manifold contained in $\mathbb{C}^2$, which is impossible. In particular, lemma 2.3 applies.

Consider a sequence $r_{1,k} \to r_1$ with $r_{1,k} \notin W^u(p_1) \cup W^s(q_1)$. Dropping to a subsequence, we may assume that $F_1^{-n_k}(r_{1,k})$ converges to a point $p_1' \in W^s(p_1) \setminus \{p_1\}$ and that $F_1^{m_k}(r_{1,k})$ converges to a point $q_1' \in W^u(q_1) \setminus \{q_1\}$ with $p_1', q_1' \in U_1$.

Let V_1^u be a compact neighborhood of r_1 in $W^u(p_1)$ and V_1^s a compact neighborhood of r_1 in $W^s(q_1)$. Let d denote Euclidean distance, and let $d_{1,k}^u = d(r_{1,k}, V_1^u)$ and $d_{1,k}^s = d(r_{1,k}, V_1^s)$, both of which converge to 0 as $k \to \infty$. Again dropping to a subsequence, we may assume that $\log d_{1,k}^s / \log d_{1,k}^u$ converges to $L \in [0, \infty]$.

Using section 7.3 of [dMvS], we see that F_1 is C^1-linearizable in neighborhoods of p_1 and q_1. Hence for some constant $c > 1$ independent of k, we have $|\alpha_1|^{n_k}/c \le d_{1,k}^u \le c|\alpha_1|^{n_k}$ and $|\beta_1|^{-m_k}|/c \le d_{1,k}^s \le c|\beta_1|^{-m_k}$. Taking log and keeping track of signs, we see that for k large,

$$\frac{n_k \log|\alpha_1| - \log c}{-m_k \log|\beta_1| + \log c} \ge \frac{\log d_{1,k}^u}{\log d_{1,k}^s} \ge \frac{n_k \log|\alpha_1| + \log c}{-m_k \log|\beta_1| - \log c}.$$

The limit of the central term exists in $[0, \infty]$ and equals $1/L$. On the other hand, the ratio of the outer two terms tends to 1 as $k \to \infty$, so the limit of each of those terms is also $1/L$. Hence

$$\lim_{k\to\infty} \frac{-m_k}{n_k} = L\frac{\log|\alpha_1|}{\log|\beta_1|}.$$

Let $V_2^s = \phi(V_1^s)$, $V_2^u = \phi(V_1^u)$, $r_{2,k} = \phi(r_{1,k})$, $d_{2,k}^s = d(r_{2,k}, V_2^s)$, and $d_{2,k}^u = d(r_{2,k}, V_2^u)$. From lemma 2.3, the order of tangency is the same for both pairs of manifolds, and we may put the tangencies in the normal form as in the proof of that lemma, so that the tangency is at the origin, the unstable manifold is the z-axis, and the stable manifold is the graph of $z \mapsto z^n$ for some $n \ge 2$.

To reach a contradiction, suppose first that $\log|\alpha_1|/\log|\beta_1| < \log|\alpha_2|/\log|\beta_2|$. We will show that then either

$$\frac{d_{1,k}^u}{d_{1,k}^s} \to 0 \quad \text{or} \quad \frac{d_{2,k}^u}{d_{2,k}^s} \to \infty \tag{3.1}$$

(or both). To see this, note that if $d_{2,k}^u/d_{2,k}^s$ is bounded from above, then there exists $C > 0$ such that $|\alpha_2|^{n_k}|\beta_2|^{m_k} \le C$. Hence

$$\frac{\log|\alpha_2|}{\log|\beta_2|} \le \frac{\log C}{n_k \log|\beta_2|} - \frac{m_k}{n_k}. \tag{3.2}$$

Choose $\delta > 1$ such that $(\log|\alpha_1| + \log\delta)/\log|\beta_1| = \log|\alpha_2|/\log|\beta_2|$. Using this to replace the left hand side of (3.2), we find

$$n_k \log(|\alpha_1|\delta) + m_k \log|\beta_1| \leq \frac{\log C \log|\beta_1|}{\log|\beta_2|},$$

so $(|\alpha_1|\delta)^{n_k}|\beta_1|^{m_k}$ is bounded and hence $|\alpha_1|^{n_k}|\beta_1|^{m_k} \to 0$. Given the bounds on $d^u_{1,k}$ in terms of $|\alpha_1|$ and on $d^s_{1,k}$ in terms of $|\beta_1|$, we see that $d^u_{1,k}/d^s_{1,k} \to 0$, as desired. Similarly, if $\log|\alpha_1|/\log|\beta_1| > \log|\alpha_2|/\log|\beta_2|$, then either $d^u_{1,k}/d^s_{1,k} \to \infty$ or $d^u_{2,k}/d^s_{2,k} \to 0$.

Let $A^s = \{(z,w) : |w - z^n| \leq |z|^n/8\}$, $A^u = \{(z,w) : |w| \leq |z|^n/8\}$, and let A^0 be the closure of the complement of $A^s \cup A^u$. Note that for $r = (z,w)$ near r_1, $d(r, V^s_1) \approx |w - z^n|$, and $d(r, V^u_1) \approx |w|$, both up to a constant multiple which approaches 1 as r tends to r_1. In particular, if $d^u_{1,k}/d^s_{1,k} \to 0$, then $r_{1,k} \in A^u$ for large k, while if $d^u_{1,k}/d^s_{1,k} \to \infty$, then $r_{1,k} \in A^s$ for large k. Hence, if $r_{1,k} \in A^0 \cup A^s$ for all k, then $d^u_{1,k}/d^s_{1,k}$ is bounded below by a positive constant, while if $r_{1,k} \in A^0 \cup A^u$ for all k, then $d^u_{1,k}/d^s_{1,k}$ is bounded above by a positive constant. Thus (3.1) implies that if $\log|\alpha_1|/\log|\beta_1| < \log|\alpha_2|/\log|\beta_2|$, then there exists a neighborhood B_1 of r_1 such that $\phi(B_1 \cap (A^0 \cup A^s)) \subseteq A^s$. Likewise, if $\log|\alpha_1|/\log|\beta_1| > \log|\alpha_2|/\log|\beta_2|$, then there exists B_1 with $\phi(B_1 \cap (A^0 \cup A^u)) \subseteq A^u$.

To show the first case is impossible, let $\gamma_1(t) = (0, \epsilon e^{it})$. Then $\gamma_1 \subseteq B_1 \cap A^0$ for small ϵ, and γ_1 is a generator for $\pi_1(\mathbb{C} \times \mathbb{C}^*)$. If $\phi \circ \gamma_1(t) \subseteq A^s$, we can project along fibers parallel to the w-axis to V^s_2 to get a curve γ_2. Note that γ_2 misses the origin since the point of tangency is preserved, so γ_2 projects to the z-axis to give an element $[m] \in \pi_1(\mathbb{C}^*)$. Since V^s_2 is the graph of $z \mapsto z^n$ for some $n \geq 2$, we see that $\gamma_2 = [mn] \in \pi_1(\mathbb{C} \times \mathbb{C}^*)$, so γ_2 is not a generator and hence $\phi \circ \gamma_1$ is not a generator, contradiction. Thus $\phi(B_1 \cap (A^0 \cup A^s)) \not\subseteq A^s$.

To show the second case is impossible, apply the map $(z,w) \mapsto (z, z^n - w)$ to both systems. This interchanges A^s and A^u, so the preceding argument applies to show that $\phi(B_1 \cap (A^0 \cup A^u)) \not\subseteq A^u$. Thus $\log|\alpha_1|/\log|\beta_1| = \log|\alpha_2|/\log|\beta_2|$ as desired. ∎

Proof of theorem 1.2: First we observe that by [B1], there exists $N \in \mathbb{Z}^+$ such that if $d \geq N$, then there is an open set $\mathcal{U} \subseteq \mathcal{P}_d$ and a dense subset $\mathcal{E} \subseteq \mathcal{U}$ such that each $F \in \mathcal{E}$ has a hyperbolic fixed point $p = p(F)$ with a homoclinic tangency: a tangency between the stable and unstable manifolds of p.

Fix $F \in \mathcal{U}$, and let $\mathcal{V}$ be any neighborhood of F in $\mathcal{P}_d$. It suffices to find F_1 and F_2 in $\mathcal{V}$ which are not conjugate.

Choose $G \in \mathcal{V}$ with a homoclinic tangency associated to a fixed point p, and consider the family G_a defined in proposition 2.4. If there exists $G_a \in V$ with no homoclinic tangencies associated to a fixed point, then we are done by lemma 2.3 since tangencies are preserved under conjugacy.

Otherwise, let α and β be the contracting and expanding eigenvalues of D_pG, respectively. By lemma 3.1, it suffices to show that there is a μ near 1 such that G_μ has no fixed point whose eigenvalues $\alpha(\mu)$ and $\beta(\mu)$ satisfy $\log|\alpha(\mu)|/\log|\beta(\mu)| = \log|\alpha|/\log|\beta|$. By proposition 2.4, in any neighborhood of 1 there exists an open set, U, of parameters μ such that each G_μ has d distinct fixed points, each of which has distinct associated eigenvalues which are nonconstant holomorphic functions of μ. Let $p(\mu)$ be such a fixed point with contracting and expanding eigenvalues $\alpha(\mu)$ and $\beta(\mu)$, respectively. Suppose $\log|\alpha(\mu)|/\log|\beta(\mu)|$ equals a constant C_1

for some open set of μ's. Since $\alpha(\mu)\beta(\mu)$ equals JG_μ, which is a constant C_2 by proposition 2.4, we can replace $\beta(\mu)$ by $C_2/\alpha(\mu)$ to obtain $\log|\alpha(\mu)| = C_1 \log|C_2|/(1+C_1)$. This implies that $|\alpha(\mu)|$ and hence $\alpha(\mu)$ are constant, contradiction. Thus $\log|\alpha(\mu)|/\log|\beta(\mu)|$ is a nonconstant real-analytic function on U, so there is an open dense subset on which this function is not equal to $\log|\alpha|/\log|\beta|$. Taking the intersection of these open dense subsets corresponding to the d distinct fixed points of G_μ shows that we can choose μ near 1 as claimed. Hence F is not stable. ■

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