Part I. Questions in number 1 and 2 are short questions. You do not need to provide details. No partial credits are given in this part except 1 (e) and 2.

1. Answer the following questions according to the given matrices:

\[ A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}, \quad A^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 3 \\ 0 & 1 \\ 1 & -2 \end{bmatrix}. \]

(a) (3 pts) \((A^t)^{-1} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}\). True or False.

True

(b) (3 pts) \(A^t \cdot B^t = (A \cdot B)^t\). True or False.

False

(c) (3 pts) Let \(M = B \cdot A\). The linear transformation \(T_M\) is invertible. True or False.

False

\(B \cdot A\) is not square

(d) (3 pts) Choose a correct statement: (Circle your answer.)

i. \(B\) describes a linear transformation from \(\mathbb{R}^3\) to \(\mathbb{R}^2\).

ii. \(B\) describes a linear transformation from \(\mathbb{R}^2\) to \(\mathbb{R}^3\).

(e) (4 pts) Compute the product of matrices \(A^{-1} \cdot B^t \cdot B \cdot A\) if possible.

\[ \begin{bmatrix} -3 & -100 \\ 9 & 39 \end{bmatrix} \]
2. (5 pts) Each of the following (a), (b), (c), (d) provides a set of vectors in \( \mathbb{R}^4 \). Determine if the vectors in each set form a basis for \( \mathbb{R}^4 \). Circle the set(s) in which vectors form a basis. If none of them does, circle NONE here.

(a) \( \{[1, 0, 3, 0]^t, [1, 2, 1, 1]^t, [2, 1, -5, 8]^t\} \).
These vectors are linearly independent.

(b) \( \{v_1 = [2, 3, 1, 2]^t, v_2 = [5, 2, 1, 2]^t, v_3 = [1, -4, -1, -2]^t, v_4 = [11, 0, 1, 2]^t\} \).
It is known that \( 4v_1 - 4v_2 + v_3 + v_4 = 0 \).

(c) \( \{[1, 3, 1, 2]^t, [0, 3, 1, 1]^t, [1, 2, 1, 0]^t, [2, 8, 3, 3]^t\} \). The reduced row echelon form of the matrix
\[
\begin{bmatrix}
1 & 3 & 1 & 2 \\
0 & 3 & 1 & 1 \\
1 & 2 & 1 & 0 \\
2 & 8 & 3 & 3 \\
\end{bmatrix}
\]
has no zero row.

(d) \( \{[1, -1, 3, 4]^t, [2, 0, -2, 1]^t, [3, 1, 0, -1]^t, [0, 1, 1, 5]^t, [1, 3, -2, 2]^t\} \).
Part II. Give comprehensive explanations for the rest of questions. Some questions might have short answers. You must indicate how your answer comes from.

3. For a given matrix $A$ as the following, we call the reduced row echelon form of $A$ as $R$,

$$
A = \begin{bmatrix}
1 & 1 & 1 & -1 & 4 \\
2 & 4 & 0 & 0 & 4 \\
2 & 6 & -2 & 2 & 0 \\
0 & -1 & 1 & 0 & 0
\end{bmatrix}, \quad \text{and } R = \begin{bmatrix}
1 & 0 & 2 & 0 & 2 \\
0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 1 & -2 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}.
$$

(a) (5 pts) Find a basis of the row space of $A$. (Answer directly.)

$$
\begin{align*}
\mathbf{u}_1 & = \begin{bmatrix} 1 & 0 & 2 & 0 & 2 \end{bmatrix} \\
\mathbf{u}_2 & = \begin{bmatrix} 0 & 1 & -1 & 0 & 0 \end{bmatrix} \\
\mathbf{u}_3 & = \begin{bmatrix} 0 & 0 & 0 & 1 & -2 \end{bmatrix}
\end{align*}
$$

form a basis for the row space of $A$.

(b) (5 pts) What is Nullity($A$)?

$$
\text{rank } A = 3
$$

$$
\text{Nullity } (A) = \text{ # of columns in } A - \text{ rank } A = 4
$$

(c) (5 pts) Write the third row vector in $A$ as a linear combination of the basis elements answered in Part (a).

$$
\begin{bmatrix} 2 & 6 & -2 & 2 & 0 \end{bmatrix} = 2 \mathbf{u}_1 + 6 \mathbf{u}_2 - 2 \mathbf{u}_3
$$

( Check! )

(d) (5 pts) Do columns of $R$ span the column space of $A$? Explain.

No!

The column space of $A$ has a basis as \{ $\begin{bmatrix} 1/3 \\ 0 \\ -1/3 \\ 0 \\ 1/3 \end{bmatrix}, $ $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ \}$

These vectors do not generate all the columns in $A$,

$$
\begin{bmatrix} 4 \\ 2 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ -2 \\ 4 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \notin \text{ Span } \{ $\begin{bmatrix} 1/3 \\ 0 \\ -1/3 \\ 0 \\ 1/3 \end{bmatrix}, $ $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ \}$
4. The matrix \( A = \begin{bmatrix} 1 & -2 & 3 & 1 \\ 2 & -4 & 1 & 1 \\ 4 & -8 & 7 & 3 \end{bmatrix} \) describes a linear transformation \( T_A : \mathbb{R}^4 \rightarrow \mathbb{R}^6 \).

(a) (12 pts) Find a basis for the image of \( T_A \).

The image of \( T_A \) = the column space of \( A \).

\[ A = \begin{bmatrix} 1 & -2 & 3 & 1 \\ 2 & -4 & 1 & 1 \\ 4 & -8 & 7 & 3 \end{bmatrix} \xrightarrow{\text{Row Reduction}} \begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 0 & -5 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

\[ \sim \begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 0 & 1 & \frac{1}{5} \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{Row Reduction}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & \frac{1}{5} \\ 0 & 0 & 1 & 0 \end{bmatrix} \]

\( T_A : \mathbb{R}^4 \rightarrow \mathbb{R}^3 \), \( \begin{bmatrix} \frac{1}{2} \\ \frac{3}{4} \end{bmatrix} \), \( \begin{bmatrix} 1 \\ 3 \end{bmatrix} \) form a basis for the image of \( T_A \).

(b) (10 pts) Since \( A \) is not a square matrix, \( T_A \) is not an invertible linear transformation. Does \( T_A \) fail to be one-to-one or onto or both? Explain why.

\( \circ \) The nullity of \( A = 2 \), or the target space has smaller dimension than domain does. These explains \( T_A \) is not 1-1.

\( \circ \) The rank of \( A = 2 \) \( \Rightarrow \) dim(image of \( T_A \)) = 2.

\( \Rightarrow \) The image of \( T_A \) is a 2-dimensional subspace in \( \mathbb{R}^3 \).

Therefore, \( T_A \) is not onto.
5. (a) (12 pts) Determine if the matrix \( M = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 2 \\ -1 & -1 & -1 \end{bmatrix} \) has an inverse matrix. Write down the inverse explicitly if it exists. (Suggestion: Check your answer if time is permitted to avoid unnecessary mistakes.)

\[
M' \text{ exists and } M' = \begin{bmatrix} 0 & -1 & -2 \\ -1 & 2 & 0 \\ 1 & -1 & 1 \end{bmatrix}
\]

(b) (10 pts) Find the solution for the linear system:

\[
\begin{align*}
2x + 3y + 4z &= -2 \\
x + 2y + 2z &= 4 \\
x - y - z &= -1.
\end{align*}
\]

This linear system has the coefficient matrix as \( M \) in (a), which is invertible.

\[
\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 2 \\ -1 & -1 & -1 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -10 \\ -7 \\ 1 \end{bmatrix}
\]

Or, one may use the augmented matrix and take more time to do it.
(c) (5 pts) Continue from the previous page. Does the vector $\begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}$ belong to the column space of $M$?

The column space of $M$ is the image of $T_M$.

Since $M$ is invertible, let $\omega_X = \begin{bmatrix} x \\ y \end{bmatrix}$.

then $M \cdot X = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is always solvable for any vector in $\mathbb{R}^3$.

$\begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}$ is in the image of $T_M$ and therefore, belongs to the column space of $M$.

6. (10 pts) What matrix describes the linear transformation that rotates $\mathbb{R}^2$ clockwise by $\frac{\pi}{2}$ radian and shortens the magnitude by one half? (Suggestion: You may start with finding the image of the unit square by using the graph. No need to use the trigonometric formula.)

$$
\begin{align*}
\begin{bmatrix} 0 \\ -1 \end{bmatrix} & \rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{rotate} \\
\begin{bmatrix} 0 \\ 1 \end{bmatrix} & \rightarrow \begin{bmatrix} 0 \\ -1 \end{bmatrix} \quad \text{shorten}
\end{align*}
$$

The matrix describes this lin. transf.

is $\begin{bmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix}$

(The columns are exactly the image of the standard basis under the lin. transf.)