Solve the initial value problem
\[ u'' + u = 0.5 \cos t, \quad u(0) = 0, \quad u'(0) = 0, \] (25)
and plot the graph of the solution.

The general solution of the differential equation is
\[ u = c_1 \cos t + c_2 \sin t + 0.25 \sin t, \]
and the initial conditions require that \( c_1 = c_2 = 0 \). Thus the solution of the given initial value problem is
\[ u = 0.25 \sin t. \] (26)

The graph of the solution is shown in Figure 3.8.8.

**FIGURE 3.8.8** Solution of Eq. (25):
\[ u'' + u = 0.5 \cos t, \quad u(0) = 0, \quad u'(0) = 0; \quad u = 0.25 \sin t. \]

Because of the term \( \sin nt \), the solution (24) predicts that the motion will become unbounded as \( t \to \infty \) regardless of the values of \( c_1 \) and \( c_2 \), and Figure 3.8.8 bears this out. Of course, in reality, unbounded oscillations do not occur, because the spring cannot stretch infinitely far. Moreover, as soon as \( n \) becomes large, the mathematical model on which Eq. (17) is based is no longer valid, since the assumption that the spring force depends linearly on the displacement requires that \( n \) be small. As we have seen, if damping is included in the model, the predicted motion remains bounded; however, the response to the input function \( F_0 \cos \omega t \) may be quite large if the damping is small and \( \omega \) is close to \( \omega_0 \).
15. Find the solution of the initial value problem

\[ u'' + u = F(t), \quad u(0) = 0, \quad u'(0) = 0, \]

where

\[ F(t) = \begin{cases} \sin t, & 0 \leq t \leq \pi, \\ \sin 2t, & \pi < t \leq 2\pi, \\ 0, & 2\pi < t. \end{cases} \]

Hint: Treat each time interval separately, and match the solutions in the different intervals by requiring \( u \) and \( u' \) to be continuous functions of \( t \).

16. A series circuit has a capacitor of \( 0.25 \times 10^{-6} \text{ F} \), a resistor of \( 5 \times 10^3 \text{ Ω} \), and an inductor of \( 1 \text{ H} \). The initial charge on the capacitor is zero. If a 12-volt battery is connected to the circuit and the circuit is closed at \( t = 0 \), determine the charge on the capacitor at \( t = 0.001 \text{ s} \), at \( t = 0.01 \text{ s} \), and at any time \( t \). Also determine the limiting charge as \( t \rightarrow \infty \).

17. Consider a vibrating system described by the initial value problem

\[ u'' + \frac{1}{4} u' + 2u = 2 \cos \omega t, \quad u(0) = 0, \quad u'(0) = 2. \]

(a) Determine the steady state part of the solution of this problem.
(b) Find the amplitude \( A \) of the steady state solution in terms of \( \omega \).
(c) Plot \( A \) versus \( \omega \).
(d) Find the maximum value of \( A \) and the frequency \( \omega \) for which it occurs.

18. Consider the forced but undamped system described by the initial value problem

\[ u'' + u = 3 \cos \omega t, \quad u(0) = 0, \quad u'(0) = 0. \]

(a) Find the solution \( u(t) \) for \( \omega \neq 1 \).
(b) Plot the solution \( u(t) \) versus \( t \) for \( \omega = 0.7 \), \( \omega = 0.8 \), and \( \omega = 0.9 \). Describe how the response \( u(t) \) changes as \( \omega \) varies in this interval. What happens as \( \omega \) takes on values closer and closer to 1? Note that the natural frequency of the undamped system is \( \omega = 1 \).

19. Consider the vibrating system described by the initial value problem

\[ u'' + u = 3 \cos \omega t, \quad u(0) = 1, \quad u'(0) = 1. \]

(a) Find the solution for \( \omega \neq 1 \).
(b) Plot the solution \( u(t) \) versus \( t \) for \( \omega = 0.7 \), \( \omega = 0.8 \), and \( \omega = 0.9 \). Compare the results with those of Problem 18; that is, describe the effect of the nonzero initial conditions.

20. For the initial value problem in Problem 18, plot \( u' \) versus \( u \) for \( \omega = 0.7 \), \( \omega = 0.8 \), and \( \omega = 0.9 \). Such a plot is called a phase plot. Use \( \pi \) interval that is long enough so that the phase plot appears as a closed curve. Mark your curve with arrows to show the direction in which it is traversed as \( t \) increases.

21 through 23 deal with the initial value problem

\[ u'' + 0.125u' + 4u = F(t), \quad u(0) = 2, \quad u'(0) = 0. \]

In each of these problems:
(a) Plot the given forcing function \( F(t) \) versus \( t \), and also plot the solution \( u(t) \) versus \( t \) on the same set of axes. Use a \( \pi \) interval that is long enough so the initial transients are substantially eliminated. Observe the relation between the amplitude and phase of the forcing term and the amplitude and phase of the response. Note that \( \omega_0 = \sqrt{4} = 2 \).

(b) Draw the phase plot of the solution; that is, plot \( u' \) versus \( u \).

REFERENCES


There are many books on mechanical vibrations and electric circuits. One that deals with both is

A classic book on mechanical vibrations is

A more recent, intermediate-level book is

An elementary book on electric circuits is