LESSON 10

Chapter 7.2 Trigonometric Integrals

1. Basic trig integrals you should know.
   \[ \int \sin x \, dx = - \cos x + C \]
   \[ \int \cos x \, dx = \sin x + C \]
   \[ \int \sec^2 x \, dx = \tan x + C \]
   \[ \int \sec x \tan x \, dx = \sec x + C \]
   \[ \int \csc^2 x \, dx = - \cot x + C \]
   \[ \int \csc x \cot x \, dx = - \csc x + C \]
   \[ \int \tan x \, dx = \ln |\sec x| + C = - \ln |\cos x| + C \]
   \[ \int \sec x \, dx = \ln |\sec x + \tan x| + C \]
   \[ \int \cot x \, dx = \ln |\sin x| + C \]
   \[ \int \csc x \, dx = \ln |\csc x - \cot x| + C \]

2. Products and powers of trig functions.

   Usually the idea is to change the integral to the form \( \int u^n \, du \) (or the integral of linear combinations of \( u^n \)).

   Integrands containing powers of sine and cosine functions. \( \int \sin^m x \cos^n x \, dx \).
   - Odd powers of \( \sin x \) – factor odd factor of \( \sin x \) and use the identity \( \sin^2 x = 1 - \cos^2 x \) to convert remaining even powers of \( \sin x \) to \( \cos x \). Then let \( u = \cos x \).
   - Odd powers of \( \cos x \) – factor odd factor of \( \cos x \) and use the identity \( \cos^2 x = 1 - \sin^2 x \) to convert remaining even powers of \( \cos x \) to \( \sin x \). Then let \( u = \sin x \).
   - No odd powers of \( \sin x \) or \( \cos x \) – use \( \cos^2 x = \frac{1}{2} + \frac{1}{2} \cos(2x) \) or \( \sin^2 x = \frac{1}{2} - \frac{1}{2} \cos(2x) \).
     After integrating the trig identity \( \sin 2x = \sin x \cos x \) may be useful.

   Integrands containing tangent and secant functions. \( \int \tan^m x \sec^n x \, dx \).
   - factor \( \sec^2 x \) and change remaining factors of \( \sec^2 x \) to tangent using \( \sec^2 x = \tan^2 x + 1 \).
   - factor \( \tan x \sec x \) and change remaining factors of \( \tan^2 x \) to secant using \( \tan^2 x = \sec^2 x - 1 \).
LESSONS 11 and 12

Chapter 7.3 Trig Substitution

If integrand contains*  Use this substitution  And use this identity
\[ \sqrt{a^2 - (u(x))^2} \quad u(x) = a \sin \theta \quad 1 - \sin^2 \theta = \cos^2 \theta \]
\[ \sqrt{a^2 + (u(x))^2} \quad u(x) = a \tan \theta \quad 1 + \sin^2 \theta = \sec^2 \theta \]
\[ \sqrt{(u(x))^2 - a^2} \quad u(x) = a \sec \theta \quad \sec^2 \theta - 1 = \tan^2 \theta \]

* Sometimes the integrand will contain powers of these roots.

Completing the square is sometimes necessary to get the forms \( a^2 - (u(x))^2 \), \( a^2 + (u(x))^2 \), or \( (u(x))^2 - a^2 \).

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Chapter 7.3 Partial Fractions

• Let \( \frac{P(x)}{Q(x)} \) be a rational function (\( P(x) \) and \( Q(x) \) are both polynomials) where \( P \) and \( Q \) have no common factors and the degree of \( P \) is less than the degree of \( Q \). If \( P \) and \( Q \) have common factors, cancel them. If degree of \( P \) not less than degree of \( Q \), then use long division to divide \( P \) by \( Q \).

• Factor the denominator \( Q(x) \) into linear factors \((ax+b)\) and irreducible quadratic factors \((ax^2+bx+c\), where \(b^2-4ac<0)\).

• \( m \) linear factors \((ax+b)^m\) give rise to \( m \) partial fractions of the form
\[
\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \frac{A_3}{(ax+b)^3} + \cdots + \frac{A_{m-1}}{(ax+b)^{m-1}} + \frac{A_m}{(ax+b)^m}
\]
\(A_1, A_2, A_3, \ldots, A_n\) are constants. When you set up your partial fractions use \( A, B, C, D, \ldots \) in the numerators instead. With the examples we use, you won’t run out of letters!

• \( n \) irreducible quadratic factors \((ax^2+bx+c)^n\) give rise to \( n \) partial fractions of the form
\[
\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \frac{A_3x+B_3}{(ax^2+bx+c)^3} + \cdots + \frac{A_{n-1}x+B_{n-1}}{(ax^2+bx+c)^{n-1}} + \frac{A_nx+B_n}{(ax^2+bx+c)^n}
\]
\(A_1, A_2, A_3, \ldots, A_n\) and \(B_1, B_2, B_3, \ldots, B_n\) are constants. When you set up your partial fractions use \(Ax+B, Cx+D, Ex+F, \ldots\) in the numerators instead. With the examples we use, you won’t run out of letters!

Set the rational function \( \frac{P(x)}{Q(x)} \) equal to its sum of partial fractions. Next multiply both sides of this equation by the common denominator \( Q(x) \). Then solve for the constants by either choosing values of \( x \) that will eliminate all but one constant, and/or equating the coefficients of like powers of \( x \) to get a system of linear equations where the unknowns are the constants. Solve that system using your ”favorite” method.
• When integrating the partial fractions, you will usually be dealing with the following kinds of integrals:

\[
\int \frac{1}{x-a} \, dx = \ln |x-a| + C \\
\int \frac{1}{(x-a)^n} \, dx = \frac{(x-a)^{n+1}}{n+1} + C, \quad n \neq -1 \\
\int \frac{1}{x^2 + a} \, dx = \frac{1}{\sqrt{a}} \tan^{-1} \frac{x}{\sqrt{a}} + C 
\]

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Chapter 7.6 Integration using tables

When using a table of integrals to evaluate \( \int f(x) \, dx \), remember that \( x \) is not always equal to \( u \). Sometimes you first must use a substitution \( u = u(x) \). If you don’t, your answer will be incorrect by a constant factor.

Chapter 7.7 Approximate Integration

To approximate \( \int f(x) \, dx \), subdivide the interval \([a, b]\) into \( n \) equal subintervals. Let \( x_i = a + i \left( \frac{b-a}{n} \right) \). Note that \( x_0 = a \) and \( x_n = b \).

- **Midpoint Rule:**
  \[
  \int_a^b f(x) \, dx \approx M_n = \left( \frac{b-a}{n} \right) \left( f(x_0 + x_1) + f \left( \frac{x_1 + x_2}{2} \right) + f \left( \frac{x_2 + x_3}{2} \right) + \cdots + f \left( \frac{x_{n-1} + x_n}{2} \right) \right)
  \]

- **Trapezoidal Rule:**
  \[
  \int_a^b f(x) \, dx \approx T_n = \left( \frac{b-a}{2n} \right) (f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)).
  \]

- **Simpson’s Rule:** **Note: \( n \) must be even**
  \[
  \int_a^b f(x) \, dx \approx S_n = \left( \frac{b-a}{3n} \right) (f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)).
  \]
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Chapter 7.8 Improper Integrals

The integral \( \int_{a}^{b} f(x)dx \) is "improper" if at least one of the following is true:

- \( a = \infty \)
- \( b = -\infty \)
- \( f \) has an infinite discontinuity at some point \( c \) in the interval \([a, b]\) (the graph of \( f \) has a vertical asymptote \( x = c \)).

We determine whether an "improper" integral has a value by evaluating the limit of "proper" integrals as follows: (remember to evaluate the proper integral on the right-hand side before evaluating the limit)

**Improper integral of Type I**

- \( \int_{a}^{\infty} f(x)dx = \lim_{t \to \infty} \int_{a}^{t} f(x)dx. \)
- \( \int_{-\infty}^{b} f(x)dx = \lim_{t \to -\infty} \int_{t}^{b} f(x)dx. \)

**Improper integral of Type II**

- If \( f \) has a discontinuity at \( a \), then \( \int_{a}^{b} f(x)dx = \lim_{t \to a^+} \int_{t}^{b} f(x)dx. \)
- If \( f \) has a discontinuity at \( b \), then \( \int_{a}^{b} f(x)dx = \lim_{t \to b^-} \int_{a}^{t} f(x)dx. \)

If the limit exists, then the limit value is the value of the improper integral. If the limit does not exist, then the improper integral is divergent.

If \( f \) has an infinite discontinuity at \( c \) where \( a < c < b \), then

\[
\int_{a}^{b} f(x)dx = \int_{a}^{c} f(x)dx + \int_{c}^{b} f(x)dx
\]

provided BOTH improper integrals on the right hand side exist. If either diverges, then \( \int_{a}^{b} f(x)dx \) diverges.

**An important integral**: \( \int_{1}^{\infty} \frac{1}{x^p} dx \) is convergent if \( p > 1 \) and divergent if \( p \leq 1 \).

**Comparison Test for Improper Integrals** When you can’t determine an anti-derivative for an improper integral, a comparison with a known convergent or divergent integral is useful.

- Suppose that \( f \) and \( g \) are continuous functions with \( 0 \leq g(x) \leq f(x) \) for \( x \geq a \).
If \( \int_a^\infty f(x) \, dx \) is convergent, then \( \int_a^\infty g(x) \, dx \) is convergent.

If \( \int_a^\infty g(x) \, dx \) is divergent, then \( \int_a^\infty f(x) \, dx \) is divergent.

LESSON 17

Chapter 8.1 Arc length

- If \( y = f(x) \) and \( a \leq x \leq b \), then \( L = \int_a^b \sqrt{1 + (f'(x))^2} \, dx \)
- If \( x = g(y) \) and \( c \leq y \leq d \), then \( L = \int_c^d \sqrt{1 + (g'(y))^2} \, dy \)

Chapter 8.2 Area of a surface of revolution

Rotate about the \( x \)-axis

\[ R = \text{radius} = f(x) \]

Surface Area = \( \int_a^b 2\pi f(x) \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx \)

Rotate about the \( y \)-axis

\[ R = \text{radius} = x \]

Surface Area = \( \int_a^b 2\pi x \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx \)