No notes, calculators, or other electronic devices such as cell phones are allowed during the exam. Violation of this policy will result in an automatic 0 on the test. There should be nothing on your desk other than the test and something to write with. Please put all work and answers on the test sheets. If extra space is needed, please the backs of the pages. Show all work. A correct answer without supporting work is worth NO credit!

Please note that there are some matrices and their echelon forms at the end of the test.

(1) Bring the following matrix into row reduced echelon form: 
\[
\begin{bmatrix}
-5 & -10 & -10 & -5 & 5 & 5 \\
3 & 6 & 7 & 3 & 3 & 3 \\
1 & 2 & 2 & 1 & 0 & 0
\end{bmatrix}
\]

(2) Let \( A, X \) and \( B \) be as below.
\[
A = \begin{bmatrix}
1 & 2 & 8 & 5 \\
-2 & -1 & -7 & -4 \\
3 & -2 & 0 & -1
\end{bmatrix}
\quad B = \begin{bmatrix}
-4 \\
2 \\
4
\end{bmatrix}
\quad X = \begin{bmatrix}
x \\
y \\
z \\
w
\end{bmatrix}.
\]

Find the general solution to the system \( AX = B \) expressed in terms of translation and spanning vectors.

(3) For which values of \( a, b, \) and \( c \) will the following system have a solution?
\[\begin{align*}
x + 2y + z + w &= a \\
x - z - 3w &= b \\
y + z + 2w &= c
\end{align*}\]

(4) Create a 3 × 4 matrix \(A\) with all of its entries > 0 such that \(AX = B\) is solvable if and only if \(B\) belongs to the span of \(U\) and \(V\) where

\[
U = \begin{bmatrix}
1 \\
-2 \\
-2 \\
3
\end{bmatrix}, \quad V = \begin{bmatrix}
1 \\
1 \\
1 \\
1
\end{bmatrix}.
\]

Explain why your answer works.

(5) Let

\[
X_1 = [1, 1, 1, 1]^t, \quad X_2 = [2, 1, 3, 2]^t, \quad X_3 = [4, 3, 5, 4]^t
\]

\[
Y_1 = [1, 0, 2, 1]^t, \quad Y_2 = [3, 2, 4, 3]^t.
\]

Prove prove that \(X_1, X_2,\) and \(X_3\) span the same subspace of \(\mathbb{R}^3\) as \(Y_1\) and \(Y_2\).

(6) Let \(W\) be the set of all 3 × 3 matrices of the form shown below where \(a, b,\) and \(c\) range over all real numbers. Prove that none of the subspace properties((1)-(3) in Theorem 6 on p. 79 of the text) hold for \(W\).

\[
\begin{bmatrix}
1 & a & b \\
0 & 1 & c \\
0 & 0 & 1
\end{bmatrix}
\]

(7) Demonstrate your understanding of the test for independence by using it to test the following matrices for independence. (Other methods will not be accepted.) You MUST indicate clearly how any equations or matrices you use come from the test for independence and how you reach your conclusions.

(a) \[A = \begin{bmatrix}
1 & 2 \\
1 & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
4 & 1 \\
-1 & 1
\end{bmatrix}, \quad C = \begin{bmatrix}
-5 & -3 \\
0 & -1
\end{bmatrix}\]

5 pts

(b) \[
\begin{bmatrix}
2 \\
1 \\
3 \\
-1
\end{bmatrix}, \quad \begin{bmatrix}
1 \\
1 \\
4 \\
-3
\end{bmatrix}, \quad \begin{bmatrix}
1 \\
2 \\
1 \\
2
\end{bmatrix}, \quad \begin{bmatrix}
2 \\
1 \\
1 \\
1
\end{bmatrix}
\]

5 pts
(8) Find a basis for the column space of $A$ consisting of columns of $A$ where

$$
A = \begin{bmatrix}
-2 & 4 & -1 & -4 & -1 \\
3 & -6 & -2 & -1 & -9 \\
2 & -4 & 0 & 2 & -2 \\
-3 & 6 & 1 & -1 & 6 \\
-1 & 2 & -3 & -7 & -8
\end{bmatrix}
$$

(9) Let $A$ be the matrix in Problem 8. Find a basis for the column space of $A$ with the property that each basis element has a one in a position where all of the other elements have a 0.

(10) Let $A$ be the matrix in Problem 8. **Without doing any row reduction** prove that $X = [1,1,1,1]^t$ does NOT belong to the column space of $A$. *Hint:* Use one of the bases found in Problems 8 and 9.

(11) Find a basis for the nullspace of the matrix $A$ in Problem 8.

(12) $A$ is a $4 \times 6$ matrix. You are given that the following vectors form a basis for the nullspace of $A$:

$$X_1 = [2, 1, 2, 3, 0, 3]^t$$
$$X_2 = [-1, 2, 1, 3, 3, -1]^t$$

(a) What is the rank of $A$? Explain.

(b) Prove that $AZ \neq 0$ where $Z = [1,1,1,1,1,1]^t$.

(c) Is there a $X \in \mathbb{R}^6$ such that $AX = [2,1,2,3]^t$? Explain.
(d) Suppose that there is a \( Y \in \mathbb{R}^6 \) such that \( AY = [1, 1, 1]^t \).

Is there only one such \( Y \)? Explain.

2 pts

(13) Let \( W \) be the set of all \( 2 \times 3 \) matrices of the following form where \( a, b, \) and \( c \) range over all real numbers.

\[
\begin{bmatrix}
 a + 2c & a + b + 3c & 2a + b + 5c \\
 b + c & a + b + 3c & 3a + 6c
\end{bmatrix}
\]

4 pts

(a) Find a set of matrices that spans \( W \).

(b) How does it follow from part 13a that \( W \) is a subspace of \( M(2, 3) \)?

1 pts

2 pts

(c) What is the dimension of \( W \)? Explain.

(14) Suppose \( V \) is a vector space and \( \{X_1, X_2, X_3\} \) is a basis for \( V \). Let \( Y_1 = X_1 + 2X_2, \ Y_2 = 2X_1 - X_3 \) and \( Y_3 = X_2 - X_3 \).

4 pts

(a) Prove that \( \{Y_1, Y_2, Y_3\} \) is linearly independent.
(b) Prove that \{Y_1, Y_2, Y_3\} is a basis for \( V \).

(15) Let \{Y_1, Y_2, Y_3\} be a 3 element set in a 2-dimensional vector space \( V \). Using only the definition of dimension and results about systems of equations prove that the set \{Y_1, Y_2, Y_3\} is linearly dependent. You are not allowed to use any theorems about dimension in the proof. Your proof should follow the proof of Theorem 1 in Section 2.2

Remark. This result was proved in the text as part of our discussion of Theorem 1 in Section 2.2. All of the other dimension theorems were based on Theorem 1. This is why you are not allowed to use other results about dimension in your proof.

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Appendix: Some matrices \( A \) and their reduced forms \( R \).

\[
A = \begin{bmatrix}
1 & 2 & 8 & 5 & -4 \\
-2 & -1 & -7 & -4 & 2 \\
3 & -2 & 0 & -1 & 4
\end{bmatrix}
R = \begin{bmatrix}
1 & 0 & 2 & 1 & 0 \\
0 & 1 & 3 & 2 & -2 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
A = \begin{bmatrix}
1 & -2 & 3 \\
2 & -1 & -2 \\
8 & -7 & 0 \\
5 & -4 & -1 \\
-4 & 2 & 4
\end{bmatrix}
R = \begin{bmatrix}
1 & 0 & -7/3 \\
0 & 1 & -8/3 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

\[
A = \begin{bmatrix}
1 & 4 & -5 \\
2 & 1 & -3 \\
1 & -1 & 0 \\
0 & 1 & -1
\end{bmatrix}, \quad R = \begin{bmatrix}
1 & 0 & -1 \\
0 & 1 & -1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

\[
A = \begin{bmatrix}
1 & 2 & 1 & 0 \\
4 & 1 & -1 & 1 \\
-5 & -3 & 0 & -1
\end{bmatrix}
R = \begin{bmatrix}
1 & 0 & -3/7 & 2/7 \\
0 & 1 & 5/7 & -1/7 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
\[ A = \begin{bmatrix} 2 & 1 & 3 & -1 \\ 1 & 1 & 4 & -3 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 1 & 1 \end{bmatrix} \quad R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

\[ A = \begin{bmatrix} 2 & 1 & 1 & 2 \\ 1 & 1 & 2 & 1 \\ 3 & 4 & 1 & 1 \\ -1 & -3 & 2 & 1 \end{bmatrix} \quad R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

\[ A = \begin{bmatrix} -2 & 3 & 2 & -3 & -1 \\ 4 & -6 & -4 & 6 & 2 \\ -1 & -2 & 0 & 1 & -3 \\ -4 & -1 & 2 & -1 & -7 \\ -1 & -9 & -2 & 6 & -8 \end{bmatrix} \quad R = \begin{bmatrix} 1 & 0 & -4/7 & 3/7 & 11/7 \\ 0 & 1 & 2/7 & -5/7 & 5/7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]

\[ A = \begin{bmatrix} -2 & 4 & -1 & -4 & -1 \\ 3 & -6 & -2 & -1 & -9 \\ 2 & -4 & 0 & 2 & -2 \\ -3 & 6 & 1 & -1 & 6 \\ -1 & 2 & -3 & -7 & -8 \end{bmatrix} \quad R = \begin{bmatrix} 1 & -2 & 0 & 1 & -1 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]

\[ A = \begin{bmatrix} 2 & 1 & 2 & 3 & 0 & 3 \\ -1 & 2 & 1 & 3 & 3 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \quad R = \begin{bmatrix} 1 & 0 & 0 & -3/2 & 0 & 1/2 \\ 0 & 1 & 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 7/2 & -1 & 3/2 \end{bmatrix} \]

\[ A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \\ 3 & 3 & 1 \\ 0 & 3 & 1 \\ 3 & -1 & 1 \end{bmatrix} \quad R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]
\[
A = \begin{bmatrix}
1 & 2 & 0 \\
2 & 0 & 1 \\
0 & -1 & -1 \\
\end{bmatrix} \quad R = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

\[
A = \begin{bmatrix}
1 & 2 & 0 \\
2 & 0 & -1 \\
0 & 1 & -1 \\
\end{bmatrix} \quad R = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]