

11 Wednesday, February 4

Volumes of Solids of Revolution about Arbitrary Axes

Earlier we talked about revolving graphs about the \( x \) and \( y \) axes. Now suppose we want to revolve around an axis shifted away from the \( x \) and \( y \) axes

\[
A(x) = \pi [f(x) - a]^2
\]

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\]

**Theorem 11.1** (Volume of a Solid of Revolution). The volume of the solid generated by revolving about the line \( y = b \) (or \( x = b \)) the graph of \( y = f(x) \) (or the graph of \( x = f(y) \)) from \( x = c \) to \( x = d \) (or \( y = c \) to \( y = d \)) is the integral

\[
V = \int_c^d \pi [f(x) - a]^2 \, dx \quad \text{or} \quad V = \int_c^d \pi [f(y) - b]^2 \, dy
\]  

(11.1)

Similarly, for the washer method with the graphs of \( y = f(x) \) and \( y = g(x) \) (or \( x = f(y) \) and \( x = g(y) \)), the volume is the integral

\[
V = \int_c^d \pi \left( [f(x) - a]^2 - [g(x) - a]^2 \right) \, dx \quad \text{or} \quad V = \int_c^d \pi \left( [f(y) - b]^2 - [g(y) - b]^2 \right) \, dy
\]  

(11.2)
Example 11.2. Find the volume of the following solids of revolution formed by revolving the regions bounded by the given curves about the given axes.

(1) $y = x^2, y = 4, x = 0$
   
   (a) about the $x$-axis
   
   $V = \int_0^2 \pi \left( [4]^2 - [x^2]^2 \right) \, dx$

   (b) about the $y$-axis
   
   $V = \int_0^4 \pi \left( \sqrt{y} \right)^2 \, dy$

   (c) about the line $y = 4$
   
   $V = \int_0^2 \pi \left( x^2 - 4 \right)^2 \, dx$

   (d) about the line $x = -1$
   
   $V = \int_0^4 \pi \left( [\sqrt{y} - (-1)]^2 - [0 - (-1)]^2 \right) \, dy$

   $= \int_0^4 \pi \left( [1 + \sqrt{y}]^2 - 1 \right) \, dy$
(2) $y = \frac{4}{x^3}, y = 1/2, x = 1$

(a) about the $x$-axis

$$V = \int_1^3 \pi \left( \left[ \frac{4}{x^3} \right]^2 - \left[ \frac{1}{3} \right]^2 \right) \, dx$$

(b) about the $y$-axis

$$V = \int_{1/3}^4 \pi \left( \left[ \frac{4}{y} \right]^{rac{1}{3}} \right)^2 - 1^2 \right) \, dy$$

(c) about the line $x = 2$

$$V = \int_{1/3}^4 \pi \left( [1 - 2]^2 - \left[ \frac{4}{y} \right]^{rac{1}{3}} - 2 \right)^2 \, dy$$

(d) about the line $y = 4$

$$V = \int_1^3 \pi \left( \left[ \frac{1}{3} - 4 \right]^2 - \left[ \frac{4}{x^3} - 4 \right]^2 \right) \, dx$$
(3) $y = x^2 - 2x, y = 0$

(a) about the $x$-axis

$$V = \int_{0}^{a} \pi (x^2 - 2x)^2 \, dx$$

(b) about the $y$-axis

$$x = 1 \pm \sqrt{y + 1}$$

$$y + 1 = x^2 - 2x + 1 = (x-1)^2$$

$$V = \int_{-1}^{0} \pi \left( [1+\sqrt{y+1}]^3 - [1-\sqrt{y+1}]^3 \right) \, dy$$

(c) about the line $x = 2$

$$V = \int_{-1}^{0} \pi \left( [1-\sqrt{y+1}]^3 - [1+\sqrt{y+1}]^3 \right) \, dy$$

(d) about the line $y = 2$

$$V = \int_{0}^{3} \pi \left( [x^2 - 3x - a]^3 - [0-a]^3 \right) \, dx$$
Example 11.3.

(1) A pulley is shaped like the graph of \( y = 2 + \sec x \) from \( x = -\pi/4 \) to \( x = \pi/4 \) rotated about the \( x \)-axis is to have a hole drilled through the center of it (parallel to the \( x \)-axis). What should the radius of this hole be cut the original volume in half?

(2) A wok is designed to be shaped like a spherical bowl. If the radius of the sphere is is based off of is 16 cm and the bowl is 9 cm deep, what is the resulting volume?

\[
x^2 + y^3 = 256
\]
\[
x^2 = 256 - y^3
\]
\[
x = \pm \sqrt[3]{256 - y^3}
\]

\[
V = \int_{-\pi/4}^{\pi/4} \pi (256 - y^3)^{\frac{3}{2}} \, dy
\]
(3) The arch \( y = \cos x, \pi/2 \leq x \leq \pi/2 \) is revolved about the line \( y = c, 0 \leq c \leq 1 \). What is the value of \( c \) that minimizes the volume of the solid, and what is the minimum volume?

\[
y = 1 - x^2 \quad -1 \leq x \leq 1
\]

\[
c = 1 - x^2 \quad \Rightarrow \quad x = \pm \sqrt{1 - c}
\]

\[
V_1 = \int_{-1}^{-\sqrt{1-c}} \pi (1-x^2-c)^2 \, dx
\]