10 Wednesday, February 4

Reminder: Exam 1 is on Monday, February 9, 8:00pm-9:00pm in Elliott Hall. Next Friday will be a review day, and the following Monday will be an optional review day.

Volumes of Solids

Suppose we want to find the volume of a solid as shown below, and suppose we know the cross sectional area $A(x)$ at every $x$.

Approximate the volume with cylinders based on $A(x)$.

1. Divide $[a, b]$ into $n$ equal subintervals of width $\Delta x = \frac{b - a}{n}$.
2. Pick $c_i$ in each subinterval, and build a cylinder on the cross section at $x = c_i$.
3. Volume of each cylinder is $V_i = \text{base area} \times \text{height} = A(c_i) \Delta x$
4. Sum up the volumes and take limit as $n \to \infty$

$$V = \lim_{n \to \infty} \sum_{i=0}^{n-1} A(c_i) \Delta x = \int_{a}^{b} A(x) \, dx$$

**Theorem 10.1** (Volume of a solid). The volume of a solid whose cross sectional area is given by $A(x)$ from $x = a$ to $x = b$ is the integral

$$V = \int_{a}^{b} A(x) \, dx.$$  (10.1)
Example 10.2. Find the volume of the following solids.

(1) A pyramid 3 m high with a square base 3 m on a side.

(2) A solid whose base is a circle in the $xy$-plane and cross sections are squares with bases in the $xy$-plane.
Volumes of Solids of Revolution: Disk Method

Suppose that the graph of $y = f(x)$ is rotated about the $x$-axis. Then the cross sectional area of the resulting solid of revolution is that of a circle,

$$A(x) = \pi [\text{radius}]^2 = \pi [f(x)]^2$$

**Theorem 10.3** (Volume of a Solid of Revolution: Disk Method). The volume of the solid generated by revolving about the $x$-axis is the region between the $x$-axis and the graph of the continuous function $y = f(x), a \leq x \leq b$, is

$$V = \int_a^b \pi [\text{radius}]^2 \, dx = \int_a^b \pi [f(x)]^2 \, dx$$

(10.2)

**Example 10.4.** Find the volume of the following solids of revolution formed by revolving the regions bounded by the given curves about the given axis.

(1) $y = \sqrt{x}, y = 0, x = 4$ about the $x$-axis
(2) $y = x^3, y = 0, x = 2$ about the $x$-axis

(3) $y = \sec x, y = 0, x = -\pi/4, x = \pi/4$ about the $x$-axis

(4) $x = 2/y, y = 1, y = 4$ about the $y$-axis
Volumes of Solids of Revolution: Washer Method

Suppose now that the region to be revolved is bounded between two curves \( y = f(x) \) and \( y = g(x) \) with \( f(x) \geq g(x) \). Then the cross sectional area is bounded by two concentric circles, giving the area

\[
A(x) = \pi \left[ \text{outer radius} \right]^2 - \pi \left[ \text{inner radius} \right]^2 = \pi \left( [f(x)]^2 - [g(x)]^2 \right)
\]  

(10.3)

**Theorem 10.5** (Volume of a Solid of Revolution: Washer Method).

\[
V = \int_a^b \pi \left( [\text{outer radius}]^2 - [\text{inner radius}]^2 \right) \, dx = \int_a^b \pi \left( [f(x)]^2 - [g(x)]^2 \right) \, dx
\]  

(10.4)

**Example 10.6.** Find the volume of the following solids of revolution formed by revolving the regions bounded by the given curves about the given axis.

1. \( y = x^2 + 1, y = -x + 3 \) about the \( x \)-axis
(2) \( y = 1, y = \sqrt{\cos x}, x = -\pi/2, x = \pi/2 \) about the \( x \)-axis

(3) \( y = x^2, y = 2x \) in the first quadrant about the \( y \)-axis

(4) \( y = \sec x, y = \tan x, x = 0, x = 1 \) about the \( x \)-axis