19 Monday, March 2

Representing Functions as Power Series

As mentioned last time, there are two principal questions we want to answer about power series:

1. For what values of $x$ does a given power series converge?
2. If a power series does converge, is there a closed form expression that denotes its sum?

Today we will answer the converse of the second question: given a function, can we find a power series that converges to it? There are several methods of constructing power series from a function:

1. Using the formula for a geometric series
2. Substitution into a known power series
3. Adding/Subtracting known power series
4. Integration/Differentiation of a known power series
5. Creating a Taylor/Maclaurin series

Method (5) will be talked about on Wednesday. For now, we will focus on (1)-(4).

Using the Formula for a Geometric Series

**Theorem 19.1** (Geometric Series). The series $\sum_{n=0}^{\infty} ar^n$ behaves the following:

$$\sum_{n=0}^{\infty} ar^n = \begin{cases} \frac{a}{1-r} & |r| < 1 \\ \text{diverges} & |r| \geq 1 \end{cases}$$ (19.1)

By rewriting a function to match the expression $a/(1-r)$, we can form a power series for that function.

**Example 19.2.** Find a power series for the following functions about the given centers. Determine the radius/interval of convergence for each series.

1. $f(x) = \frac{3}{1-x}$, $c = 0$

   We want to match up $f(x)$ with the sum of a geometric series, $a/(1-r)$. Thus, if we make the comparison between the two expressions,

   $$\frac{a}{1-r} = \frac{3}{1-x} \quad \Rightarrow \quad a = 3 \quad r = x$$

   Putting these constants into the geometric series gives

   $$f(x) = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} 3x^n$$

   As for the convergence, the geometric series converges if $|r| < 1$. So make the comparison of this inequality to $|x - c| < R$ to get the following.

   $$|r| = |x| < 1 \quad \Rightarrow \quad R = 1 \quad I = (-1, 1)$$
(2) \( f(x) = \frac{5}{1 + 3x}, \ c = 0 \)

Make the comparison between \( f(x) \) and \( a/(1 - r) \),

\[
\frac{a}{1 - r} = \frac{5}{1 + 3x} \quad \Rightarrow \quad a = 5 \quad r = -3x
\]

Putting these constants into the geometric series gives

\[
f(x) = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} 5(-3x)^n = \sum_{n=0}^{\infty} 5(-3)^n x^n
\]

For the convergence, the geometric series converges if \(|r| < 1\). So make the comparison of this inequality to \(|x - c| < R\) to get the following.

\[
|r| = | -3x | < 1 \quad \Rightarrow \quad R = 1/3
\]

\[
I = (-1/3, 1/3)
\]

(3) \( f(x) = \frac{2}{4 - x}, \ c = 0 \)

Make the comparison between \( f(x) \) and \( a/(1 - r) \). Note that this time, \( f(x) \) does not have a hard-coded 1 in the denominator, but the sum formula does. Therefore, we need to alter \( f(x) \) to make it match

\[
\frac{a}{1 - r} = \frac{2}{4 - x} \cdot \frac{1/4}{1/4} = \frac{1/2}{1 - (x/4)} \quad \Rightarrow \quad a = 1/2 \quad r = x/4
\]

Putting these constants into the geometric series gives

\[
f(x) = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} \frac{1}{2} \left( \frac{x}{4} \right)^n = \sum_{n=0}^{\infty} \frac{x^n}{2^{2n+1}}
\]

For the convergence, the geometric series converges if \(|r| < 1\). So make the comparison of this inequality to \(|x - c| < R\) to get the following.

\[
|r| = |x/4| < 1 \quad \Rightarrow \quad R = 4
\]

\[
I = (-4, 4)
\]

(4) \( f(x) = \frac{6}{3 - 4x^2}, \ c = 0 \)

Make the comparison between \( f(x) \) and \( a/(1 - r) \).

\[
\frac{a}{1 - r} = \frac{6}{3 - 4x^2} \cdot \frac{1/3}{1/3} = \frac{2}{1 - (4x^2/3)} \quad \Rightarrow \quad a = 2 \quad r = 4x^2/3
\]

Putting these constants into the geometric series gives

\[
f(x) = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} 2 \left( \frac{4x^2}{3} \right)^n = \sum_{n=0}^{\infty} \frac{2^{2n+1}}{3^n} x^{2n}
\]

For the convergence, the geometric series converges if \(|r| < 1\). So make the comparison of this inequality to \(|x - c| < R\) to get the following.

\[
|r| = |4x^2/3| < 1 \quad \Rightarrow \quad |x^2| < 3/4 \quad \Rightarrow \quad |x| < \sqrt{3}/2 \quad \Rightarrow \quad R = \sqrt{3}/2
\]

\[
I = (-\sqrt{3}/2, \sqrt{3}/2)
\]
(5) \( f(x) = \frac{x}{x-1}, \ c = 0 \)

Make the comparison between \( f(x) \) and \( a/(1 - r) \).

\[
\frac{a}{1 - r} = \frac{x}{x-1} \cdot -1 = \frac{-x}{1 - x} \quad \Rightarrow \quad a = -x \quad r = x
\]

Putting these constants into the geometric series gives

\[
f(x) = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} (-x)x^n = -\sum_{n=0}^{\infty} x^{n+1}
\]

For the convergence, the geometric series converges if \(|r| < 1\). So make the comparison of this inequality to \(|x - c| < R\) to get the following.

\[
|r| = |x| < 1 \quad \Rightarrow \quad R = 1 \quad I = (-1, 1)
\]

(6) \( f(x) = \frac{5}{1 + x}, \ c = -2 \)

Make the comparison between \( f(x) \) and \( a/(1 - r) \). This time however, we are centering around \( c = -2 \), which means the power series contains powers of \((x + 2)\). Therefore, we never want to see \( x \) by itself; we want to see \((x + 2)\). We can achieve this by adding and subtracting 2 in the denominator:

\[
\frac{a}{1 - r} = \frac{5}{1 + x} \cdot \frac{1 - 1}{1 + x} = \frac{5}{1 - 2 + 2 + x} \cdot \frac{-1}{1 - (x + 2)} = \frac{-5}{1 - (x + 2)} \quad \Rightarrow \quad a = -5 \quad r = x + 2
\]

Putting these constants into the geometric series gives

\[
f(x) = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} -5(x + 2)^n
\]

For the convergence, the geometric series converges if \(|r| < 1\). So make the comparison of this inequality to \(|x - c| < R\) to get the following.

\[
|r| = |x + 2| < 1 \quad \Rightarrow \quad R = 1 \quad I = (-3, -1)
\]
Substitution into a Known Power Series

For this technique, we will replace the variable of the power series with an expression involving \( x \) (instead of just \( x \)).

**Example 19.3.** Given the following power series, find a power series for the following functions.

\[
e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad \arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}
\]

(1) \( f(x) = e^{2x-1} \)

Replace \( x \) with \( 2x - 1 \) in \( e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \).

\[
e^{2x-1} = \sum_{n=0}^{\infty} \frac{(2x-1)^n}{n!}
\]

(2) \( f(x) = \sin (x^2) \)

Replace \( x \) with \( x^2 \) in \( \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \).

\[
\sin (x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}
\]

(3) \( f(x) = \cos (x^3) \)

Replace \( x \) with \( x^3 \) in \( \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \).

\[
\cos (x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n}}{(2n)!}
\]

(4) \( f(x) = \arctan (x^6) \)

Replace \( x \) with \( x^6 \) in \( \arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \).

\[
\arctan (x^6) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^6)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{12n+6}}{2n+1}
\]
Adding/Subtracting Known Power Series

**Theorem.** If \( f(x) = \sum_{n=0}^{\infty} a_n (x - c)^n \) with radius of convergence \( R_f \) and \( g(x) = \sum_{n=0}^{\infty} b_n (x - c)^n \) with radius of convergence \( R_g \), then

\[
f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n) (x - c)^n
\]

and the radius of convergence of the new power series is \( \min(R_f, R_g) \).

**Example 19.4.** Find a power series about \( c = 0 \) for the following functions. Determine the radius/interval of convergence for each series.

(1) \( f(x) = \frac{x - 6}{x^2 - 4} = \frac{2}{x + 2} - \frac{1}{x - 2} \)

(Note: for these problems you do not need to concern yourselves with getting the decomposition into two fractions. The decomposition would be given to you.) First, we need to find power series for the two fractions \( \frac{2}{x + 2} \) and \( \frac{1}{x - 2} \) and their corresponding radii of convergence.

\[
\frac{2}{x + 2} = \frac{1}{1 + (x/2)} \implies a = 1 \quad r = -x/2 \implies \frac{2}{x + 2} = \sum_{n=0}^{\infty} \left( \frac{x}{2} \right)^n = \sum_{n=0}^{\infty} \left( -\frac{1}{2} \right)^n x^n, \quad R = 2
\]

and

\[
\frac{1}{x - 2} = \frac{1}{1 - (x/2)} \implies a = -1/2 \quad r = x/2 \implies \frac{1}{x - 2} = \sum_{n=0}^{\infty} \left( -\frac{1}{2} \right) \left( \frac{x}{2} \right)^n = \sum_{n=0}^{\infty} \left( -\frac{1}{2} \right)^n x^n, \quad R = 2
\]

Now we want the difference between the two series, so

\[
f(x) = \frac{2}{x + 2} - \frac{1}{x - 2} = \sum_{n=0}^{\infty} \left( \frac{1}{2} \right)^n x^n - \sum_{n=0}^{\infty} \frac{1}{2n+1} x^n
\]

\[
= \sum_{n=0}^{\infty} \left[ \left( \frac{1}{2} \right)^n x^n - \frac{1}{2n+1} x^n \right]
\]

\[
= \sum_{n=0}^{\infty} \left[ \left( \frac{1}{2} \right)^n - \frac{1}{2n+1} \right] x^n
\]

\[
= \sum_{n=0}^{\infty} \left[ (-1)^n + \frac{1}{2n+1} \right] \frac{x^n}{2^n}
\]

Both series that comprise this new series have a radius of convergence equal to \( R = 2 \), so this new series also has a radius of convergence of \( R = \min(2, 2) = 2 \).
(2) \( f(x) = \frac{5x + 4}{2x^2 + 3x + 1} = \frac{1}{x + 1} + \frac{3}{2x + 1} \)

First, we need to find power series for the two fractions \( \frac{1}{x + 1} \) and \( \frac{3}{2x + 1} \) and their corresponding radii of convergence.

\[
\frac{1}{x + 1} \implies a = 1 \quad r = -x \implies \frac{1}{x + 1} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n, \quad R = 1
\]

and

\[
\frac{3}{2x + 1} \implies a = 3 \quad r = -2x \implies \frac{3}{2x + 1} = \sum_{n=0}^{\infty} 3(-2x)^n = \sum_{n=0}^{\infty} 3(-2)^n x^n, \quad R = 1/2
\]

Now we want the sum of the two series, so

\[
f(x) = \frac{1}{x + 1} + \frac{3}{2x + 1} = \sum_{n=0}^{\infty} (-1)^n x^n + \sum_{n=0}^{\infty} 3(-2)^n x^n
\]

\[
= \sum_{n=0}^{\infty} \left[ (-1)^n x^n + 3(-2)^n x^n \right]
\]

\[
= \sum_{n=0}^{\infty} \left[ (-1)^n + 3(-2)^n \right] x^n
\]

The series that comprise this sum have radii of convergence \( R = 1 \) and \( R = 1/2 \), respectively. The new series has a radius of convergence that is the smaller of the two, \( R = \min(1, 1/2) = 1/2 \).
Integration/Differentiation of a Known Power Series

**Theorem 19.5.** Let \( f(x) = \sum_{n=0}^{\infty} a_n (x - c)^n \) be a power series with radius of convergence \( R \). Then \( f \) has derivatives of all orders, computed by differentiating term by term:

\[
\begin{align*}
  f'(x) &= \sum_{n=1}^{\infty} n a_n (x - c)^{n-1} \\
  f''(x) &= \sum_{n=2}^{\infty} n(n-1) a_n (x - c)^{n-2} \\
  &\vdots
\end{align*}
\]

In addition, each of these derivative series have a radius of convergence \( R \). \( f \) also can be integrated term by term:

\[
\int f(x) \, dx = C + \sum_{n=0}^{\infty} \frac{a_n (x - c)^{n+1}}{n + 1}
\]

and this integrated series also has a radius of convergence \( R \).

**Example 19.6.** Find a power series about \( c = 0 \) for the following functions. Determine the radius/interval of convergence for each series.

(1) \( f(x) = \ln(x + 1) \)

To find the power series for this function, we will exploit the fact that

\[
\ln(x + 1) = C + \int \frac{dx}{x + 1}
\]

The power series for \( 1/(x+1) \) is (see previous example)

\[
\frac{1}{x+1} = \sum_{n=0}^{\infty} (-1)^n x^n, \quad R = 1
\]

Now substitute this series for \( 1/(x+1) \) in the integral.

\[
\ln(x+1) = C + \int \left[ \sum_{n=0}^{\infty} (-1)^n x^n \right] dx
\]

What the above theorem says is that we can interchange the summation with the integral; that is,

\[
\ln(x + 1) = C + \int \left[ \sum_{n=0}^{\infty} (-1)^n x^n \right] dx = C + \sum_{n=0}^{\infty} \left[ \int (-1)^n x^n \, dx \right]
\]

Now evaluate the integral like usual, treating \( n \) as a constant. Thus,

\[
\ln(x + 1) = C + \sum_{n=0}^{\infty} \left[ \int (-1)^n x^n \, dx \right] = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n + 1}
\]

Notice that in the power series, we see a lot of \( n + 1 \). Typically, we will want to reindex the integrated series to remove that.

\[
\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n + 1} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}
\]
Now, the constant $C$. Evaluation of this series at $x = 0$ returns 0 because every term has an $x$ in it (it started with power 1 on $x$). Therefore, since $\ln(0 + 1) = \ln 1 = 0$, then $C = 0$ and

$$\ln(x + 1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n}$$

Lastly, the series $1/(x + 1)$ that we integrated to get this new series for $\ln(x + 1)$ had a radius of convergence of $R = 1$. The theorem stated that after integration, the new series has the same radius of convergence. Therefore, this series for $\ln(x + 1)$ has $R = 1$.

(2) $f(x) = \frac{1}{(2 - x)^2}$

To find the power series for this function, we will exploit the fact that

$$\frac{1}{(2 - x)^2} = \frac{d}{dx} \left( \frac{1}{2 - x} \right)$$

The power series for $1/(2 - x)$ is (see example 19.4(1))

$$\frac{1}{2 - x} = \sum_{n=0}^{\infty} \frac{x^n}{2^{n+1}}, \quad R = 2$$

Now substitute this series for $1/(2 - x)$ in the derivative.

$$\frac{1}{(2 - x)^2} = \frac{d}{dx} \left[ \sum_{n=0}^{\infty} \frac{x^n}{2^{n+1}} \right]$$

What the above theorem then says is that we can interchange the summation with the derivative; that is,

$$\frac{1}{(2 - x)^2} = \frac{d}{dx} \left[ \sum_{n=0}^{\infty} \frac{x^n}{2^{n+1}} \right] = \sum_{n=0}^{\infty} \left[ \frac{d}{dx} \left( \frac{x^n}{2^{n+1}} \right) \right]$$

Now evaluate the derivative like usual, treating $n$ as a constant. Thus,

$$\frac{1}{(2 - x)^2} = \sum_{n=0}^{\infty} \left[ \frac{d}{dx} \left( \frac{x^n}{2^{n+1}} \right) \right] = \sum_{n=0}^{\infty} \frac{nx^{n-1}}{2^{n+1}}$$

Notice that in this derivative series, the $n = 0$ evaluates to 0 (because the $n$ was brought down from the power of $x$). This is the usual case with derivatives of power series, so as shown in the theorem, we will adjust the index of the series by 1.

$$\frac{1}{(2 - x)^2} = \sum_{n=0}^{\infty} \frac{nx^{n-1}}{2^{n+1}} = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{2^{n+1}}$$

(This is different from reindexing the series. Reindexing a series involves rewriting the formula for the $n$th term, whereas here we know that every time we put $n = 0$, we get 0, so we are just telling ourselves to just not bother with $n = 0$ by changing it to $n = 1$).

Now to make matters worse, we will now perform a reindexing back to $n = 0$, but now its the traditional reindexing technique where we replace $n$ with $n + 1$.

$$\frac{1}{(2 - x)^2} = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{2^{n+1}} = \sum_{n=0}^{\infty} \frac{(n + 1)x^n}{2^{n+2}}$$

Lastly, the theorem said that the radius of convergence of this new series is the same as $1/(2 - x)$, so $R = 2$ for this new series.