Chapter 3

Rotations and reflections in the plane

We want another important source of nonabelian groups, which is one that most people should already be familiar. Let

\[ A = \begin{bmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots \end{bmatrix} \]

be an \( n \times n \) matrix with entries in \( \mathbb{R} \). If \( B \) is another \( n \times n \) matrix, we can form their product \( C = AB \) which is another \( n \times n \) matrix with entries

\[ c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj} = \sum_k a_{ik}b_{kj} \]

The identity matrix

\[ I = \begin{bmatrix} 1 & 0 & \cdots \\ 0 & 1 & \cdots \\ \vdots \end{bmatrix} \]

has entries

\[ \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \]

**Lemma 3.1.** Matrix multiplication is associative and \( I \) is the identity for it, i.e. \( AI = IA = A \).

**Proof.** Given matrices \( A, B, C \), the \( ij \)th entries of \( A(BC) \) and \( (AB)C \) both work out to

\[ \sum_k \sum_\ell a_{ik}b_{\ell j}c_{\ell j} \]

Also

\[ a_{ij} = \sum_k a_{ik}\delta_{kj} = \sum_k \delta_{ik}a_{kj} \]
An $n \times n$ matrix $A$ is *invertible* if there exists an $n \times n$ matrix $A^{-1}$ such that $AA^{-1} = A^{-1}A = I$. It follows that:

**Theorem 3.2.** The set of invertible $n \times n$ matrices with entries in $\mathbb{R}$ forms a group called the general linear group $GL_n(\mathbb{R})$.

For $2 \times 2$ matrices there is a simple test for invertibility. We recall that the determinant

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

and

$$e \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ea & eb \\ ec & ed \end{bmatrix}$$

**Theorem 3.3.** Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be matrix over $\mathbb{R}$, then $A$ is invertible if and only $\det(A) \neq 0$. In this case,

$$A^{-1} = (\det(A))^{-1} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

*Proof.* Let $\Delta = \det(A)$, and let $B = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. Then an easy calculation gives

$$AB = BA = \Delta I.$$

If $\Delta \neq 0$, then $\Delta^{-1}B$ will give the inverse of $A$ by the above equation.

Suppose that $\Delta = 0$ and $A^{-1}$ exists. Then multiply both sides of the above equation by $A^{-1}$ to get $B = \Delta A^{-1} = 0$. This implies that $A = 0$, and therefore that $0 = AA^{-1} = I$. This is impossible. \hfill $\Box$

Let us study an important subgroup of this. A $2 \times 2$ rotation matrix is a matrix of the form

$$R(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

This sends a column vector $v$ in the plane $\mathbb{R}^2$ to the vector $R(\theta)v$ obtained by rotation through angle $\theta$. We denote the set of these by $SO(2)$ ($SO$ stands for special orthogonal).

**Theorem 3.4.** $SO(2)$ forms a subgroup of $GL_2(\mathbb{R})$. 

\hfill $\Box$
Proof. It is easy to check that \( \det R(\theta) = \cos^2 \theta + \sin^2 \theta = 1 \) and of course, \( R(0) = I \in SO(2) \). If we multiply two rotation matrices

\[
R(\theta)R(\phi) = \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}
\begin{bmatrix}
\cos \phi & -\sin \phi \\
\sin \phi & \cos \phi
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\cos \theta \cos \phi - \sin \theta \sin \phi & - \cos \theta \sin \phi - \sin \theta \cos \phi \\
\sin \theta \cos \phi + \cos \theta \sin \phi & \cos \theta \cos \phi - \sin \theta \sin \phi
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\cos(\theta + \phi) & -\sin(\theta + \phi) \\
\sin(\theta + \phi) & \cos(\theta + \phi)
\end{bmatrix}
\]

\[
= R(\theta + \phi)
\]

Therefore \( SO(2) \) is closed under multiplication. The last calculation also shows that \( R(\theta)^{-1} = R(-\theta) \in SO(2) \)

A matrix

\[
A = \begin{bmatrix}
a & b \\
c & d
\end{bmatrix}
\]

is called orthogonal if the columns are unit vectors \( a^2 + c^2 = b^2 + d^2 = 1 \) which are orthogonal in the sense that the dot product \( ab + cd = 0 \). Since the first column is on the unit circle, it can be written as \( (\cos \theta, \sin \theta)^T \) (the symbol \((-)^T\), read transpose, turns a row into a column). The second column is on the intersection of the line perpendicular to the first column and the unit circle. This implies that the second column is \( \pm (\cos \theta, \sin \theta)^T \). So either \( A = R(\theta) \)
or

\[
A = F(\theta) = \begin{bmatrix}
\cos \theta & \sin \theta \\
\sin \theta & -\cos \theta
\end{bmatrix}
\]

In the exercises, you will find a pair of nonzero orthogonal vectors \( v_1, v_2 \), \( F(\theta)v_1 = v_1 \) and \( F(\theta)v_2 = -v_2 \). This means that \( F(\theta) \) is a reflection about the line spanned by \( v_1 \). In the exercises, you will also prove that

\textbf{Theorem 3.5.} The set of orthogonal matrices \( O(2) \) forms a subgroup of \( GL_2(\mathbb{R}) \).

Given a unit vector \( v \in \mathbb{R}^2 \) and \( A \in O(2) \), \( Av \) is also a unit vector. So we can interpret \( O(2) \) as the full symmetry group of the circle, including both rotations and reflections.

### 3.6 Exercises

1. Let \( UT(2) \) be the set of upper triangular matrices

\[
\begin{bmatrix}
1 & a \\
0 & 1
\end{bmatrix}
\]

Show this forms a subgroup of \( GL_2(\mathbb{R}) \).
2. Let UT(3) be the set of upper triangular matrices
\[
\begin{bmatrix}
1 & a & b \\
0 & 1 & c \\
0 & 0 & 1
\end{bmatrix}
\]
Show this forms a subgroup of GL_3(\mathbb{R}).

3. Find a pair of nonzero orthogonal vectors \(v_1, v_2\), \(F(\theta)v_1 = v_1\) and \(F(\theta)v_2 = -v_2\). (Hint: if \(\theta = 0\) this is easy; when \(\theta \neq 0\), try \(v_1 = (\sin \theta, 1 - \cos \theta)^T\).)

4. Recall that the transpose of an \(n \times n\) matrix \(A\) is the \(n \times n\) matrix with entries \(a_{ji}\). A matrix is called orthogonal if \(A^T A = I = AA^T\) (the second equation is redundant but included for convenience).
   (a) Check that this definition of orthogonality agrees with the one we gave for 2 \(\times\) 2 matrices.
   (b) Prove that the set of \(n \times n\) orthogonal matrices \(O(n)\) is a subgroup of GL_\(n\)(\(\mathbb{R}\)). You’ll need to know that \((AB)^T = B^T A^T\).

5. Show that SO(2) is abelian, but that O(2) is not.

6. A 3 \(\times\) 3 matrix is called a permutation matrix, if it can be obtained from the identity \(I\) by permuting the columns. Write \(P(\sigma)\) for the permutation matrix corresponding to \(\sigma \in S_3\). For example,
\[
F = P((12)) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]
Check that \(F^2 = I\). What can you conclude about the set \(\{I, F\}\)?

7. Prove that the set of permutations matrices in GL_3(\mathbb{R}) forms a subgroup. Prove the same thing for GL_\(n\)(\(\mathbb{R}\)), where permutations matrices are defined the same way. (The second part is not really harder than the first, depending how you approach it.)