Chapter 18

Counting Problems involving Symmetry*

Group theory can be applied to counting problems involving symmetry. Here are a few such problems.

**Example 18.1.** How many dice can be constructed by labeling the face of a cube by the numbers 1, . . . , 6?

**Example 18.2.** Suppose 3 identical decks of 52 cards are combined into a big deck. How many 3 card hands can be dealt out of the big deck?

**Example 18.3.** How many ways can a necklace be constructed with 2 black and 2 white beads?

To analyze the first problem, let’s first keep track of the order in which cards are dealt. Let’s suppose that the kinds of cards are labeled by numbers 1, . . . , 52. Since we have three decks we can be confident about not running out of kinds. Thus the set of ordered hands can be identified with

\[ H = \{(a, b, c) \mid a, b, c = 1, 2, \ldots, 52\} \]

The cardinality of this set is 52^3. Of course, we want to disregard order. As first guess, we might think that we should divide this by the number of ways of permuting the cards to get 52^3/6. Unfortunately, this is not an integer so it doesn’t make sense. To explain the correct answer, we will introduce some more group theory.

**Definition 18.4.** We say that a group \((G, \ast, e)\) acts on a set \(X\) if there is an operation \(\cdot : X \times G \to X\) satisfying:

1. \(x \cdot e = x\).
2. \(x \cdot (g \ast h) = (x \cdot g) \cdot h\)
(We’re are really defining a right action here, there is also a notion of left action, but we won’t need that.)

**Definition 18.5.** Given a group $G$ acting on a set $X$, the orbit of $x \in X$ is the set $x \cdot G = \{x \cdot g \mid g \in G\}$. The set of orbits is denoted by $X/G$.

**Definition 18.6.** An element $x$ is called a fixed point of $g$ is $x \cdot g = x$.

Returning to example 18.2. $S_3$ acts on $H$ by moving the positions of the cards. For example,

$$f = \begin{cases} 1 \to 2 \\ 2 \to 3 \\ 3 \to 1 \end{cases}$$

then $(a_1, a_2, a_3) \cdot f = (a_2, a_3, a_1)$. We want to treat two hands as the same if you can permute one to get the other, i.e. if they lie in the same orbit. Therefore our problem is to count the set of orbits $H/S_3$. If the first two cards are repeated, so $a_1 = a_2$, then $(a_1, a_2, a_3)$ is a fixed point for $(12)$.

**Theorem 18.7 (Burnside).** If $G$ is a finite group acting on a finite set $X$, then

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} (\# \text{ of fixed points of } g)$$

This will be proved in the next chapter.

**Corollary 18.8.** Suppose that every element other than the identity has no fixed points, then $|X/G| = |X|/|G|$.

Now, we can solve the problems mentioned earlier. For problem 18.1, we first choose an initial labelling, or marking, of our blank cube. This allows us to talk about the first face, second face and so on. Let $X$ be the set of labellings of the faces of this marked cube. There are 6 choices for the first face, 5 for the second..., therefore there are $6! = 720$ elements of $X$. $G = C$ is the symmetry group for the cube which has order 24. $G$ acts by rotating the cube. Clearly there are no fixed points for any rotation (other than $I$). Therefore the number of labellings for a blank cube is

$$|X/G| = \frac{720}{24} = 30$$

Next consider problem 18.2. Let $H_{ij}$ be the set of ordered triples $(a_1, a_2, a_3)$ with $a_i = a_j$. This is the set of fix points of $(ij)$. Since there only two free choices, we have

$$|H_{ij}| = 52^2$$

Let $H_{123}$ be the set of triples where all the cards are the same. This is the set of fixed points for $(123)$ and $(132)$. Clearly

$$|H_{123}| = 52$$
Therefore

\[
\frac{H}{S_3} = \frac{1}{6} \left( |H| + |H_{12}| + |H_{13}| + |H_{23}| + |H_{123}| + |H_{123}| \right)
\]

\[
= \frac{1}{6} \left[ 52^3 + 3(52^2) + 2(52) \right]
\]

\[
= 24804
\]

Finally, consider problem 18.3. Let us first arrange the beads in sequence. There are 6 possibilities

\[
\begin{array}{cccccc}
\bigcirc & \bigcirc & \bullet & \bullet & \bullet & \bigcirc \\
\bullet & \bullet & \bigcirc & \bigcirc & \bullet & \bigcirc \\
\bigcirc & \bullet & \bigcirc & \bigcirc & \bullet & \bigcirc \\
\end{array}
\]

and we let \(X\) be the set of these. The group in this problem is the symmetry group of the square \(D_4 \subset S_4\) which permutes the positions of the beads. All of these are fixed by the identity. The rotations \((1234)\) and \((1432)\) have no fixed points. The element \((12)(34)\) has two fixed points, namely the leftmost sequences in the above diagram. The remaining 4 elements also have two fixed points a piece (exercise). Therefore

\[
|X/D_4| = \frac{1}{8} [6 + 5(2)] = 2
\]

### 18.9 Exercises

1. Complete the analysis of problem 18.3.

2. How many necklaces can be constructed using 4 different colored beads?

3. In how many ways can the sides of a tetrahedron be labelled by the numbers 1, 2, 3, 4?

4. Suppose 2 identical decks of 52 cards are combined into a big deck. How many 3 card hands can be delt out of the big deck?

In the above calculations, certain numbers occurred multiple times. This can be explained with the help of the following definition. Two elements \(g_1, g_2 \in G\) are **conjugate** if \(g_2 = h' * g_1 * h\) for some \(h \in G\).

1. Prove that in \(S_3\), \((12)\) is conjugate to \((13)\) and \((23)\), and \((123)\) is conjugate to \((132)\).

2. Suppose that a finite group \(G\) acts on a set \(X\). Prove that if \(g_1\) and \(g_2\) are conjugate, then the number of fixed points of \(g_1\) and \(g_2\) are the same.
Chapter 19

Proofs of theorems about group actions

We first prove a strengthened version of theorem 17.3. Given a group acting a set \(X\), the stabilizer of \(x\) is

\[
\text{stab}(x) = \{ g \in G \mid x \cdot g = x \}
\]

**Theorem 19.1.** Let \(G\) be a finite group acting on a set \(X\), then \(|G| = |\text{stab}(x)||xG|

**Proof.** For each \(y \in xG\) let

\[
T(y) = \{ g \in G \mid x \cdot g = y \}
\]

Choose \(g_0 \in T(y)\), then the function \(f(g) = g \cdot g_0\) maps \(\text{stab}(x) \rightarrow T(y)\). This is a one to one correspondence since it has an inverse \(f^{-1}(h) = h \cdot g_0^{-1}\). This implies that \(T(y) = |\text{stab}(x)|\).

If \(y \neq z\) then \(T(y)\) and \(T(z)\) must be disjoint, otherwise \(y = x \cdot g = z\) for \(g \in T(y) \cap T(z)\). Every \(g\) lies in some \(T(y)\) namely \(T(x \cdot g)\). Therefore \(T(y)\) is a partition of \(G\). By corollary 2.3,

\[
|G| = \sum_{y \in xG} |T(y)| = |xG||\text{stab}(x)|
\]

\[\square\]

Given a subgroup \(H\) of a group \(G\), a (right) coset is a set of the form \(H \cdot g = \{ h \cdot g \mid h \in H \}\) for \(g \in G\). The set of cosets is denoted by \(G/H\). We can define a right action of \(G\) on \(G/H\) by

\[
(H \cdot g) \cdot \gamma = H \cdot (g \cdot \gamma)
\]

This is transitive action which means that there is only one orbit, and the stabilizer \(\text{stab}(H) = H\) (exercise). Therefore, we obtain an extension of 13.1 to nonabelian groups.
**Theorem 19.2 (Lagrange).** Let $G$ be a group of finite order. Then for any subgroup $H$, $|G| = |H||G/H|$.

We now prove Burnside’s theorem.

*Proof.* Let

$$C = \{(x, g) \in X \times G \mid x \cdot g = g\}$$

Consider the map $p : C \to G$ given by $p(x, g) = g$. Then an element of $p^{-1}(g)$ is exactly a fixed point of $g$. Therefore corollary 2.4 applied to $p$ yields

$$|C| = \sum_{g \in G} |p^{-1}(g)| = \sum_{g \in G} \text{(\# of fixed points of } g\text{)} \quad (19.1)$$

Next consider the map $q : C \to X$ given by $q(x, g) = x$. Then $q^{-1}(x) = \text{stab}(x)$. Therefore corollary 2.4 applied to $q$ yields

$$|C| = \sum_{x \in x} |p^{-1}(x)| = \sum_{x \in x} |\text{stab}(x)|$$

We group the last sum into orbits

$$C = \sum_{x \in x} \text{1st orbit} |\text{stab}(x)| + \sum_{x \in x} \text{2nd orbit} |\text{stab}(x)| + \ldots$$

For each orbit $x_0G$ has $|G|/|\text{stab}(x_0)|$ elements by theorem 19.1. Furthermore, for any $x \in x_0G$, we have $|\text{stab}(x)| = |\text{stab}(x_0)|$. Therefore

$$\sum_{x \in x_0G} |\text{stab}(x)| = \sum_{x \in x_0} |\text{stab}(x_0)| = \frac{|G|}{|\text{stab}(x_0)|} |\text{stab}(x_0)| = |G|$$

Consequently

$$|C| = |G| \sum_{\text{orbits}} 1 = |G||X/G|$$

Combining this with equation 19.1 yields

$$|G||X/G| = \sum_{g \in G} \text{(\# of fixed points of } g\text{)}$$

Dividing by $|G|$ yields the desired formula. □

### 19.3 Exercises

1. Fill in the details in the proof of Lagrange’s theorem
   a) Prove that $G$ acts transitively on $G/H$
   b) Prove that $\text{stab}(H) = H$.

2. Prove that if $G$ is a group with $|G|$ a prime, then $G$ is cyclic (compare lemma 13.5).