This small supplement contains some things discussed in class but not in the book. Given a sequence of numbers $a_0, a_1, a_2 \ldots$

its generating function is the power series

$$g(x) = a_0 + a_1 x + a_2 x^2 + \ldots$$

For our purposes, we treat this as just an expression; we won’t about convergence or things like that.

The basic example is the generating function of $1, 1, 1, \ldots$. This is the geometric series

$$f(x) = \frac{1}{1 - x} = 1 + x + x^2 + \ldots = \sum_{n=0}^{\infty} x^n \quad (1)$$

From this, we can various other examples by substituting for $x$. For example,

$$\frac{1}{1 - 2x^2} = f(2x^2) = 1 + 2x^2 + 4x^4 + \ldots = \sum_{n=0}^{\infty} 2^n x^{2n}$$

We add and multiply term by term. For example, let us evaluate

$$\frac{1}{(1 - x)^2} = (1 + x + x^2 + \ldots)(1 + x + x^2 + \ldots) \quad (2)$$

$$= (1)(1) + (1)(x) + (x)(1) + (1)(x^2) + (x)(x) + (x^2)(1) + \ldots \quad (3)$$

$$= 1 + 2x + 3x^2 + \ldots \quad (4)$$

In other words, to get $x^n$, we can pick $x^i$ from the first factor and $x^j$ from the second, with $i + j = n$. There are $n + 1$ do this, so the coefficient of $x^n$ should be $n + 1$. In general, we have

$$\frac{1}{(1 - x)^k} = (1 + x + x^2 + \ldots)(1 + x + x^2 + \ldots) \ldots (1 + x + x^2 + \ldots) = \sum_{n=0}^{\infty} a_n x^n \quad (5)$$

where $a_n$ is the number of ways to pick $x^{i_1}$ from the first factor, $x^{i_2}$ from the second and so on, such that

$$i_1 + i_2 + \ldots + i_k = n$$

1
This is a problem that we thought about before when we were doing combinations, and we have a formula for it. But even if we didn’t, we can derive in an entirely different way. Let $f(x)$ be as above. Differentiating, term by term, yields

$$f'(x) = 1 + 2x + 3x^2 + \ldots = \sum_{n=0}^{\infty} (n+1)x^n$$

On the other hand,

$$f'(x) = [(1 - x)^{-1}]' = (1 - x)^{-2}$$

This already reproves (4). Now do this again

$$f''(x) = (2)(1) + (3)(2)x + (4)(3)x^2 + \ldots = \sum_{n=0}^{\infty} (n + 2)(n + 1)x^n$$

$$f''(x) = 2(1 - x)^{-3}$$

Thus

$$\frac{1}{(1 - x)^3} = \sum_{n=0}^{\infty} \frac{(n + 2)(n + 1)}{2} x^n$$

Note that the coefficient is just \( \binom{n+2}{2} \). Doing this repeatedly gives

$$f^{(k)}(x) = \sum_{n=0}^{\infty} (n + k - 1)(n + k - 2)\ldots(n + 1)x^n$$

$$f^{(k)}(x) = (k - 1)!(1 - x)^{-k}$$

Thus

$$\frac{1}{(1 - x)^k} = \sum_{n=0}^{\infty} \binom{n + k - 1}{k - 1} x^n$$

This can be used in situations, where we don’t know the formula already (perhaps because there isn’t one). We define a partition of an positive integer $n$ as a sum $n = n_1 + n_2 + \ldots + n_k$. We want to consider two partitions the same if the differ by order e.g. $2 + 3 + 1 + 1 = 1 + 2 + 3 + 1$. One way break this ambiguity is to always write the numbers in (nonstrictly) decreasing order: $3 + 2 + 1 + 1$. We can visualize a partition by arranging numbers in a so called Young diagram. We stack $n_1$ boxes on top of $n_2$ boxes on top of $n_3$...

$p(n)$ is the number of partitions of $n$. For example,

$$4 = 4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1$$

so $p(4) = 5$. The Young diagrams look like:
Given a Young diagram

\[
\begin{array}{cccc}
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\end{array}
\]

the width \( w \) will be the maximum number of across. Let \( i_1 \) be the number of rows of consisting of 1 box across, \( i_2 \) be the number of rows of consisting of 2 boxes across and so on upto \( i_w \). It should be clear that the diagram can be reconstructed using these measurements alone. Also the total number of boxes is given by

\[
i_1 + 2i_2 + 3i_3 + \ldots wi_w = n
\tag{6}
\]

Conversely, given a (nonnegative integer) solution to this equation, we can construct a unique diagram with these measurements. Therefore the number \( p(n, w) \) of Young diagrams with \( n \) boxes and width at most \( w \) is the number of solutions to (6). Since the widest diagram with \( n \) boxes is \( n \), we have \( p(n) = p(n, n) \).

Now consider the series

\[
\frac{1}{(1-x)(1-x^2)\ldots(1-x^w)} = (1 + x + x^2 + \ldots)(1 + x^2 + x^4 + \ldots)\ldots
\tag{7}
\]

After multiplying this out, we find that the coefficient of \( x^n \) is the number of solutions to (6) which just \( p(n, w) \). Therefore

**Theorem 1.** The series (7) is the generating function \( \sum_n p(n, w)x^n \).

**Corollary 2.** The generating function of \( p(n) \) is the infinite product

\[
\frac{1}{(1-x)(1-x^2)\ldots}
\]

Using Maple (as we did in class), it easy to multiply this out to any reasonable degree:

\[
\frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)(1-x^7)(1-x^8)(1-x^9)(1-x^{10})}
\]
Thus \( p(10) = p(10, 10) = 42 \).

This technique can be used to solve many other counting problems. For example, let \( c_n \) be the number of ways that \( n \) cents can broken up as nickels, dimes and quarters. 15 cents is three nickels, or one dime and nickel, so \( c_{15} = 2 \). If \( i_1 \) is the number of nickels, \( i_2 \) the number of dimes, and \( i_3 \) the number of quarters, then

\[
5i_1 + 10i_2 + 25i_3 = n
\]

This is the number of ways for to get \( x^n \) in the expansion of

\[
(1 + x^5 + x^{10} + x^{15} + \ldots)(1 + x^{10} + x^{100} + \ldots)(1 + x^{25} + \ldots)
\]

Therefore

\[
\sum c_n x^n = \frac{1}{(1 - x^5)(1 - x^{10})(1 - x^{25})}
\]