Equivalence Relations

A relation $R$ is called an equivalence relation if it satisfies the following conditions
1. It’s reflexive, $xRx$.
2. It’s symmetric, if $xRy$ then $yRx$.
3. It’s transitive, if $xRy$ and $yRz$ then $xRz$.

Examples,
1. Equality.
2. Given the set $C = \{\text{red-shirt, blue-shirt, red-sock, yellow-sock, yellow-shirt}\}$

Define $xRy$ exactly when they have the same color.

3. Let $T$ be the set of triangles. Define $\Delta_1 \equiv \Delta_2$ if they are congruent.

In order to check 2, we make another definition. A partition of a set $S$ is a subset $P$ of the powerset $P(S)$ such that every $s \in S$ lies in exactly on subset $E \in P$. For example, grouping things of the same color

$\text{Red} = \{\text{red-shirt, red-sock}\}$

$\text{Blue} = \{\text{blue-shirt}\}$

$\text{Yellow} = \{\text{yellow-sock, yellow-shirt}\}$

yields a partition

$\{\text{Red, Blue, Yellow}\}$

of $C$.

**Lemma.** If $P$ is a partition of $S$, then define $x \sim y$ iff $x, y$ are both in the same set in $P$. Then $\sim$ is an equivalence relation.

Before, checking 3 we need to make the condition more precise. We view a triangle as a positive triple of positive real numbers $(L, K, M)$ measuring the length of the sides subject to the triangles $L + K < M, K + M < L, M + L < K$ (in English: any side is less than the sum of the other two sides).

We will say that two triangles are congruent, if these lengths are the same after relabelling the sides. Thus to each triangle, we can form the set of all possible relabelled triangles.

$\{(L, K, M), (K, M, L), (M, L, K), (L, M, K), (K, L, M), (M, K, L)\}$

This gives a partition of the set of triangles, and the corresponding equivalence relation is precisely congruence.

**Functions**

As we saw a function $f : X \rightarrow Y$ is a special kind of relation $f \subseteq X \times Y$ with the feature that for each $x \in X$, there is a unique $(x, y) \in f$. We write $y = f(x)$ in this case. As a nonmathematical example,

$l : \{\text{dog, cat, frog, mouse}\} \rightarrow \{3, 4, 5\}$

given by $l(-) =$ number of letters in the word $-$. It can be specified as a set

$l = \{(\text{dog}, 3), (\text{cat}, 3), (\text{frog}, 4), (\text{mouse}, 5)\}$

or more naturally as

$l(\text{dog}) = 3, \ldots$
Given a subset \( A \subseteq X \), we define the image of \( A \) under \( f \) to be the subset

\[
f(A) = \{ y \in Y | y = f(x) \text{ for some } x \in A \}
\]

For example, let \( d : \mathbb{N} \to \mathbb{N} \) be the function defined by \( d(x) = 2x \). Then \( d(\mathbb{N}) \) is precisely the set of even natural numbers. We say that a function \( f : X \to Y \) is onto if \( f(X) = Y \). The function \( d \) is not onto. However the function \( l : \{ \text{dog, cat, frog, mouse} \} \to \{3, 4, 5\} \) above is onto.

We say that \( f : X \to Y \) is one to one if \( f(x_1) = f(x_2) \) implies that \( x_1 = x_2 \). Equivalently, this means that \( x_1 \neq x_2 \) implies that \( f(x_1) \neq f(x_2) \). The function \( d \) above is one to one, since \( 2x_1 = 2x_2 \) implies \( x_1 = x_2 \). But \( l \) is not, since \( l(\text{cat}) = l(\text{dog}) \).

Given a function \( f : X \to Y \), the inverse is the relation

\[
f^{-1} = \{(y, x) \in Y \times X | y = f(x)\}
\]

For example,

\[
l^{-1} = \{(3, \text{dog}), (3, \text{cat}), (4, \text{frog}), (5, \text{mouse})\}
\]

Notice that this is not a function.

**Lemma.** A function \( f : X \to Y \) is one to one and onto if and only if \( f^{-1} \) is a function. In this case, \( f^{-1}(y) \) is the unique solution to \( y = f(x) \).

A one to one onto function is called a one to one correspondence.

The function \( d : \mathbb{N} \to \mathbb{N} \) above is not onto, so \( d^{-1} \) is not a function. However, if we extend this to a function \( D : \mathbb{Q} \to \mathbb{Q} \) to the set of rational numbers

\[
\mathbb{Q} = \{ \frac{a}{b} | a, b \neq 0 \text{ integers} \}
\]

by \( D(x) = 2D : \mathbb{Q} \to \mathbb{Q} \), then \( D \) is one to one and onto, and \( D^{-1}(y) = y/2 \).

**ex.** Given a function \( f : X \to Y \) and subset \( A, B \), show that \( f(A \cup B) = f(A) \cup f(B) \).

Solution:

\[
y \in f(A \cup B)
\]

\[
\leftrightarrow y = f(x) \land x \in A \cup B
\]

\[
\leftrightarrow y = f(x) \land (x \in A \lor x \in B)
\]

\[
\leftrightarrow (y = f(x) \land x \in A) \lor (y = f(x) \land x \in B)
\]

\[
\leftrightarrow (y \in f(A)) \lor (y \in f(B))
\]

\[
\leftrightarrow y \in f(A) \cup f(B)
\]

Given functions \( f : X \to Y \) and \( g : Y \to Z \), we define a new function \( g \circ f : X \to Z \) by \( g \circ f(x) = g(f(x)) \).

**Lemma.** \( g \circ f \) is one to one if \( f \) and \( g \) are both one to one.

You will show in the homework that the same thing goes for “onto”. Therefore \( g \circ f \) is a one to one correspondence if \( f \) and \( g \) are both one to one. We define a relation between sets as follows \( X \sim Y \) (read \( X \) has the same cardinality as \( Y \)) if there is a one to one correspondence \( f : X \to Y \).
Theorem. The relation $\sim$ is an equivalence relation.

Outline of proof (full details in class). In order to show that $X \sim X$, we use the identity function $i : X \rightarrow X$ given by $i(x) = x$. This is a one to one correspondence.

If $X \sim Y$, then there exists a one to one correspondence $f : X \rightarrow Y$. The inverse $f^{-1} : Y \rightarrow X$ gives a one to one correspondence in the opposite direction. Therefore $Y \sim X$.

Suppose that $X \sim Y$ and $Y \sim Z$, then there are one to one correspondences $f : X \rightarrow Y$ and $g : Y \rightarrow Z$. $g \circ f : X \rightarrow Z$ gives another one to one correspondence. Therefore $X \sim Z$.

Homework

All of the problems refers to functions $f : X \rightarrow Y$, $g : Y \rightarrow Z$, and subsets $A, B$ of $X$.

1. Prove that $f(A \cap B) = f(A) \cap f(B)$.

2. Prove that $A \subseteq B$ implies $f(A) \subseteq f(B)$.

3. Define $x_1 =_f x_2$ to mean that $f(x_1) = f(x_2)$. Prove that this is an equivalence relation.

4. Prove that $g \circ f$ is onto if $f$ and $g$ are both are.

5. Does $f(A - B) = f(A) - f(B)$? Try and prove it, and if you can’t then look for an example where this fails! (Hint: You can look at one of the above functions.)