Math 385 Handout 4: Uncountable sets

Uncountable sets

One might get the feeling that one infinite set is as big as any other, but in fact:

**Theorem.** (Cantor) The set of real numbers \( \mathbb{R} \) is uncountable.

Before giving the proof, recall that a real number is an expression given by a (possibly infinite) decimal, e.g. \( \pi = 3.141592\ldots \). The notation is slightly ambiguous since

\[
1.0 = .9999\ldots
\]

We will break ties, by always insisting on the more complicated nonterminating decimal.

**Proof** It suffices to prove that \( \mathbb{R} \) has an uncountable subsets. We work with numbers in the interval \( I = \{ x \in \mathbb{R} \mid 0 \leq x \leq 1 \} \). We give a proof by contradiction. Suppose that \( I \) was countable, and let’s say that \( f : \mathbb{N} \rightarrow I \) is a one to one correspondence, say:

\[
\begin{align*}
f(0) &= .123\ldots \\
f(1) &= .456\ldots \\
f(2) &= .789\ldots \\
&\vdots
\end{align*}
\]

Then mark the numbers down the diagonal, and construct a new number \( x \in I \) whose \( n+1 \)th decimal is different from the \( n+1 \)decimal of \( f(n) \). Then we have found a number not in the image of \( f \), which contradicts the fact \( f \) is onto.

Cantor originally applied this to prove that not every real number is a solution of a polynomial equation with integer coefficients (contrary to earlier hopes). We expand on this idea as follows. Say that a number is describable if there is a name (such as 5, \( \pi \)), or formula \( 1 + \sqrt{2}/3 \), or perhaps a computer program, for obtaining it. The point is that the description should involve a finite number of symbols in a fixed finite alphabet. Since the set of such descriptions is countable (by hw), we obtain.

**Cor.** There are real numbers which cannot be described (and in particular computed).

This is the starting point for Cantor’s theory of *transfinite* numbers. The cardinality of a countable set (denoted by the Hebrew letter \( \aleph_0 \)) is at the bottom. Then we have the cardinality of \( \mathbb{R} \) denoted by \( 2^{\aleph_0} \), because there is a one to one correspondence \( \mathbb{R} \rightarrow P(\mathbb{N}) \). Taking the powerset again leads to a new transfinite number \( 2^{2^{\aleph_0}} \). This process goes on forever thanks to:

**Theorem.** (Cantor) For any set \( X \), there does not exist a one to one correspondence from \( X \) to \( P(X) \). In particular, the power set \( P(\mathbb{N}) \) is uncountable.

**Proof** We prove this by contradiction. Suppose that \( f : X \rightarrow P(X) \) is a one to one correspondence. Define \( C = \{ x \mid x \notin f(x) \} \). Note that \( C \) is an element of \( P(X) \), it is given as \( f(y) \) for some \( y \). Either \( y \in C \) or \( y \notin C \). If \( y \in C \), then \( y \notin f(y) = C \) which is a contradiction. So \( y \notin C = f(y) \), this implies that \( y \in C \), which is again a contradiction. The only way to avoid a contradiction is for \( f \) not to exist.

**Homework**

1. Prove that the set of all irrational real numbers is uncountable.

2. Prove that the set of functions \( f : \mathbb{N} \rightarrow \mathbb{N} \) is uncountable.