Answers and some solutions for the midterm, 511 (Spring 2011)

1. Statements B and D are true. To see that D is true, notice that columns of $A^TB$ are linear combinations of columns of $A^T$, but $A^T$ has $m$ columns and $m < n$. So the dimension of the column space of $A^TB$ is at most $m$, which is strictly less than the size of $A^TB$ (which is $n \times n$).

2. A and D are subspaces.

3. B, C and E are linear.

4. The $A = LU$ factorization is

$$
\begin{pmatrix}
2 & 1 & -1 \\
-4 & -1 & 3 \\
-2 & 0 & 1
\end{pmatrix}
= 
\begin{pmatrix}
1 & 0 & 0 \\
-2 & 1 & 0 \\
-1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
2 & 1 & -2 \\
0 & 1 & -1 \\
0 & 0 & 0
\end{pmatrix}.
$$

So the column space and the row space are two-dimensional, and $N(A)$ and $N(A^T)$ have dimension one.

A basis for the column space is $(2, -4, -2)^T$, $(1, -1, 0)^T$, and a basis of $N(A)$ is $(1/2, 1, 1)^T$.

5. Gram–Schmidt gives $q_1 = (1/3)(2, 1, -2)^T$ and $q_2 = (1/\sqrt{5})(0, -2, -1)^T$.

An easy way to find the projection of the vector $c$ is to write this projection as $p(c) = c_1q_1 + c_2q_2$, where $c_1 = q_1^Tc$ and $c_2 = q_2^Tc$. The answer is $p(c) = (4/9, 20/9, 5/9)$.

A vector $x$ orthogonal to $a$ and $b$ can be found by solving the system of two linear equations $a^Tx = 0$ and $b^Tx = 0$. Then one has to normalize $x$. The answer is $q_3 = \pm(1/\sqrt{45})(-5, 2, -4)^T$ (there are two possibilities for $q_3$, pointing to the opposite directions. I asked to find any one of them).

Another way is to subtract the projection from the vector and normalize the result.

6. We want to find $a$ and $b$ which minimize

$$
\int_{-1}^{1}(ax + b - e^x)^2dx.
$$

A straightforward approach is to evaluate this integral, and then to minimize with respect to $a$ and $b$ (by taking partial derivatives, for example). There are many other ways to solve this, but here is the simplest solution, found by one student.
Notice that 1 and $x$ are orthogonal on $(-1, 1)$. So the projection of $e^x$ onto the subspace spanned by 1 and $x$ is given by the formula $a \cdot 1 + b \cdot x$, where

$$a = \frac{\int_{-1}^{1} e^x \, dx}{\int_{-1}^{1} 1^2 \, dx} = \frac{e - e^{-1}}{2},$$

and

$$b = \frac{\int_{-1}^{1} xe^x \, dx}{\int_{-1}^{1} x^2 \, dx} = \frac{3}{e}.$$

7.

$$R_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad R_2 = \begin{pmatrix} -3/5 & -4/5 \\ -4/5 & 3/5 \end{pmatrix}$$

so

$$R = R_1 R_2 = \begin{pmatrix} -4/5 & 3/5 \\ -3/5 & -4/5 \end{pmatrix},$$

which is a rotation. A product of two reflections is always a rotation.