Name:

There will be only 5 problems on the exam, each worth 10 points, some with items a), b), c), d), e), where each item is worth 2 points.

1. Determine the radii of convergence of the following series:
   a) \( \cot z = \sum_{n=0}^{\infty} a_n (z - 1)^n \),
   b) \( \sum_{n=0}^{\infty} \frac{z^n}{n!} \),
   c) \( \sum_{n=0}^{\infty} 2^n z^n \),
   d) \( \sum_{n=0}^{\infty} \frac{z^n}{n!} \),
   e) \( \cot z = \sum_{n=0}^{\infty} a_n (z - 1)^n \).

2. Find an analytic function in the complex plane, whose real part is
   \( e^{-x}(x \cos y + y \sin y) \),
   where \( z = x + iy \).

3. True or false: if \( f \) is an analytic function in a region \( D \), and \( |f'(z)| \leq 1 \) for all \( z \in D \), then \( |f(z_1) - f(z_2)| \leq |z_1 - z_2| \) for all \( z_1, z_2 \) in \( D \)? Prove it, if true, or give a counterexample, if false.

4. Suppose that \( f \) is meromorphic in the unit disc \( |z| < 1 \), and has only one simple pole \( z_0 \neq 0 \) there. Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) be the Taylor series of \( f \) at 0. Prove that
   \( z_0 = \lim_{n \to \infty} \frac{a_n}{a_{n+1}} \).

5. Find and classify all isolated singularities of the following functions in \( \mathbb{C} \).
   (Singularities at \( \infty \) should be considered too, if they are isolated. For poles, tell their multiplicities).
   \( \frac{1}{e^z - 1} - \frac{1}{\sin z}, \frac{\sin z}{z}, \frac{z}{1 - \cos z}, \sin \tan z \).
6. Evaluate the integral
\[ \int_{|z|=15} \frac{z^3 dz}{z^5 - z + 1}. \]

7. Find all solutions of the equation
\[ \sin z = 5i \]
and sketch them in the complex plane.

8. For functions \( f(z) = \sum_{n=0}^{\infty} a_n z^n \), analytic in the unit disc \( |z| < 1 \), prove
a) that \( f \) is even, if and only if \( a_n = 0 \) for all odd integers \( n \).
b) if \( f(x) \) is real for all \( x \in (-1,1) \), then \( f(z) = \overline{f(z)} \), for all \( z \) in the unit disc.

9. Suppose that \( f \) is an analytic function in the whole complex plane, which satisfies \( f(z + 1) \equiv f(z) \), and \( f(z + i) \equiv f(z) \). Prove that this \( f \) is constant.

10. Let \( u \) be a non-constant harmonic function in the whole complex plane. Prove that the set \( \{ z : u(z) = 0 \} \) is unbounded.

11. Which of the following interpolation problems are solvable for analytic functions in \( |z| < 2 \)? Here \( n = 1, 2, 3, \ldots \)
a) \( f(1/n) = (-1)^n \).
b) \( f(1/n) = (-1)^n / n \).
c) \( f(1/n) = (-1)^n n^{-2} \).
d) \( f(1/n) = n / (n + 1) \).

12. Prove that any two disjoint circles on the Riemann sphere can be mapped onto concentric circles by a fractional-linear transformation.