3. **Formal power series** are just sequences of complex numbers, with operations of addition and multiplication defined in the following way. If \( F = (a_n) \) and \( G = (b_n) \) then \( F + G = (a_n + b_n) \) and \( FG = (c_n) \), where

\[
c_n = a_0b_n + a_1b_{n-1} + \ldots + a_nb_0 = \sum_{k=0}^{n} a_nb_{n-k}.
\] (1)

3.1 Power series with these operations form a commutative ring without divisors of zero. (This ring is actually a \( \mathbb{C} \)-algebra, as it contains a copy of \( \mathbb{C} \)).

It is a custom to write a power series \( F = (a_n) \) as an infinite sum

\[
F = a_0 + a_1z + a_2z^2 + \ldots = \sum_{n=0}^{\infty} a_nz^n,
\] (2)

which is a good mnemonic for the rules of addition and multiplication (1): they are analogous to those for polynomials. The standard notation for the ring of formal power series “of letter \( z \)” is \( \mathbb{C}[[z]] \). Sometimes a power series (2) is called a *generating function* of the sequence \( (a_n) \), and the numbers \( a_0, a_1 \ldots \) are called *coefficients* of \( F \).

We stress that the signs + and \( \Sigma \) have in general no meaning for infinite sums. Neither the symbols \( z^n \) have any meaning in (2), this is just a bookkeeping devise for application of addition and multiplication rules. In particular these \( z^n \) permit us to omit those members of a sequence which are equal to 0, so we can write \( 0 + 1z + 2z^2 + 0z^3 + 0z^4 + \ldots \) simply as \( z + 2z^2 \).

3.2 Verify the identities for power series:

\[
(1 - z)(1 + z + z^2 + z^3 + \ldots) = 1,
\]

\[
(1 - z)^{-2}(1 + 2z + 3z^2 + 4z^3 + \ldots) = 1,
\]

Find the product

\[
(1 + z + z^2 + z^3 + \ldots)(1 + 2z + 3z^2 + 4z^3 + \ldots).
\]

3.3 A power series has multiplicative inverse if and only if \( a_0 \neq 0 \).

3.4 Let \( a_n \) be the number of ways to pay \( n \) cents using the US coins, less than \$1 each, and \( F \) is the generating function given by (2). Then

\[
F(z) = (1 - z)^{-1}(1 - z^5)^{-1}(1 - z^{10})^{-1}(1 - z^{25})^{-1}(1 - z^{50})^{-1}.
\]
3.5* Find $a_{100}$.

3.6 The derivative of a power series $F = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \ldots$ is defined as the power series $dF/dz = a_1 + 2a_2 z + 3a_3 z^2 + \ldots$. Show that the usual properties hold: $d(F + G)/dz = dF/dz + dG/dz$ and $d(FG)/dz = (dF/dz)G + F(dG/dz)$.

If we denote by $F(0)$ the “constant term” $a_0$ of a series (2), then coefficients can be expressed as

$$a_n = \frac{1}{n} F^{(n)}(0).$$

3.7 The order of a power series is defined as the index of the first (leftmost) non-zero coefficient. It is denoted by ord $S$. Verify that the function $S \mapsto \exp(-\text{ord } S)$ is a norm, that is satisfies a), b) and c) from 1.5. In fact, a stronger property that c) holds:

$$|x + y| \leq \max\{|x|, |y|\}.$$

Norms with such property are called for historical reasons non-Archimedean. (An Archimedean norm is one which does not have this property; all norms we encountered before were Archimedean).

3.8 Verify that $\mathbb{C}[[z]]$ is complete with respect to this norm, and that polynomials are dense in $\mathbb{C}[[z]]$. Describe in words, what convergence of a sequence means with respect to this norm.

Suppose that $F$ and $G$ are formal power series and ord $G > 0$. Then one can substitute $G$ into $F$: if $F$ is given by (2) then

$$F \circ G = \sum_{n=0}^{\infty} a_n G^n.$$

Notice that ord $G^n \geq n$, so the coefficients of $F \circ G$ are expressed as finite sums, which makes this definition possible.

3.9 This operation of composition is associative.

3.10 A formal series $F$ has a compositional inverse $F^{-1}$ iff $F(0) = 0$ and $F'(0) \neq 0$. Thus power series with these two properties make a group, whose identity element is $F(z) = z$. This group is called the group of (formal) parameters; the reasons for this name will be seen later.
A formal **Laurent series** is an expression of the form

$$\sum_{n=m}^{\infty} a_n z^n,$$

where $m$ is an integer, may be negative. The set of all Laurent series of the letter $z$ is denoted by $C((z))$. It contains $C[[z]]$ and the operations of addition, multiplication and differentiation have natural extension from $C[[z]]$ to $C((z))$.

3.11 $C((z))$ is a field. It is the quotient field of the ring $C[[z]]$.

The order function $\text{ord}$ has a natural extension to $C((z))$, so that $\exp(-\text{ord } F)$ still has all properties of a norm, thus $C((z))$ is a normed field. One can substitute a formal local parameter into a formal Laurent series.

The coefficient $a_{-1}$ in (3) plays a special role, so it has a special name, the **residue** of the series (3), and a notation $a_{-1} =: \text{res } F$.

**Theorem.** If $F$ is a formal Laurent series, and $G$ a formal local parameter, then

$$\text{res } F = \text{res } \{ (F \circ G)G' \}.$$

**Proof.** Evidently res is a linear functional, so it is enough to check the formula for the case $F(z) = z^m$. Suppose first that $m \neq -1$. Then

$$\text{res } \{ F \circ G \} G' = \text{res } G^m G' = \text{res } \left( \frac{1}{m+1} (G^{m+1})' \right) = 0,$$

because the residue of a derivative is always zero. Now for $m = -1$ we denote coefficients of $G$ by $b_n$ and obtain

$$\text{res } G^{-1} G' = \left( (b_1)^{-1} z^{-1} + \ldots \right) (b_1 + \ldots) = 1.$$

This proves the theorem.  

As an application of this theorem, we can write an explicit formula for the coefficients of a compositional inverse of a power series.

**Corollary** (Bürmann–Lagrange Formula). Let $F \in C[[z]]$, and $G$ a parameter. Then for $n > 0$, the $n$-th coefficient of $F \circ G^{[-1]}$ is equal to

$$\frac{1}{n} \text{res } (F' G^{-n}) = \frac{1}{n!} \frac{d^{n-1}}{dz^{n-1}} \left( F' \left\{ \left( \frac{z}{G} \right) \right\} \right).$$
Proof. This coefficient is
\[
\text{res } \left( z^{-n-1} F \circ G^{-1} \right) = \text{res } (G^{-n-1} F G')
\]
\[
= -\frac{1}{n} \text{res } \left( F (G^{-n})' \right) = \frac{1}{n} \text{res } (F' G^{-n}).
\]

3.12 As an example, find a closed formula for coefficients of the compositional inverse to the power series
\[
G(z) = ze^{-z} = z - z^2 + \frac{z^3}{2!} - \ldots.
\]