4. **Summable sequences.** A sequence $a = (a_n)$ of complex numbers (or more generally, elements of a Banach space) is called *summable* if

$$N = \sup \left\{ \sum_{n \in E} |a_n| : S \text{ is a finite set} \right\} < \infty.$$ 

4.1 All summable sequences form a vector space, and $N$ is a norm in this vector space. This vector space is complete, and it is called $\ell_1$.

For each summable sequence, the sequence of its partial sums $(s_k)$,

$$s_k = \sum_{n=0}^{\infty} a_n, \quad k = 0, 1, 2 \ldots$$

is a Cauchy sequence, so it has a limit. This limit is called “the sum of the series”

$$\sum_{n=0}^{\infty} a_n.$$  \hspace{1cm} (1)

Such series (whose terms form a summable sequence) are also called *absolutely convergent.*

4.2 Suppose that $n \mapsto m(n)$ is arbitrary permutation of integers (that is a bijection of the set of non-negative integers onto itself). If (1) is absolutely convergent, then

$$s^* = a_m(0) + a_m(1) + \ldots = \sum_{n=0}^{\infty} a_{m(n)}$$

is also absolutely convergent and has the same sum.

4.3 Sum is a linear functional of norm 1 on $\ell_1$.

5. **Uniform convergence.** Let $(f_n)$ be a sequence of functions, defined on some set $E$, with values in a Banach space. We say that this sequence is convergent to a function $f$ *uniformly on $E$ if for every $\epsilon > 0$ there exists an integer $N$ such that for every $x \in E$ and every $n \geq N$ we have $|f_n(x) - f(x)| < \epsilon$.*

5.1 Uniform limit of continuous functions (defined on a topological space $E$) is continuous.
5.2 Suppose that the common domain of our functions is the segment \([0,1]\), and that they are continuous. If \(f_n \rightarrow f\) uniformly then

\[
\int f_n(x) dx \rightarrow \int f(x) dx, \quad n \rightarrow \infty.
\]

5.3 Given a set \(E \subset \mathbb{C}\) we consider the Banach space \(C(E)\) of all continuous complex-valued functions on \(E\) with the sup-norm. Then the absolute convergence of a series in \(C(E)\) implies uniform on \(E\) convergence of its partial sums. Absolute convergence of such functional series in \(C(E)\) is usually called normal convergence on \(E\).

Sometimes normally convergent series are called “uniformly and absolutely convergent”, which is not precise:

5.4 Find an example of a series

\[
\sum_{n=0}^{\infty} f_n
\]

of continuous functions on \([0,1]\), such that for every \(x \in [0,1]\) the series \(\sum_{n=0}^{\infty} f_n(x)\) is absolutely convergent, and the sequence of partial sums of (2) is uniformly convergent on \([0,1]\), but (2) is not normally convergent.

6. Convergent power series. If \(F \in \mathbb{C}[[z]]\), and \(z_0\) is a complex number, we can substitute \(z_0^n\) for \(z^n\) and obtain a series of numbers, which may be absolutely convergent or not. Similarly, considering \(z^n\) as a function defined on some set \(E \subset \mathbb{C}\), we obtain a series in the Banach space \(C(E)\).

6.1 Theorem. For every \(F \in \mathbb{C}[[z]]\) there exists \(R \in [0, +\infty]\) with the following properties:

a) for each \(r \in [0, R)\) the series is normally convergent on the set \(\{z : |z| \leq r\}\), and

b) for each \(z\) such that \(|z| > R\) the series \(F(z)\) is divergent.

Proof. Let \(R = \sup \{r \geq 0 : (|a_n| r^n)\) is a bounded sequence \}. Then evidently b) holds. If \(R = 0\) then a) is void. If \(R > 0\), we take arbitrary \(r \in (0, R)\) and put \(r_1 = (R + r)/2\). Then for \(|z| \leq r\) we have

\[
|a_n z^n| \leq \left(\frac{r}{r_1}\right)^n |a_n| r_1^n \leq \text{const} \left(\frac{r}{r_1}\right)^n,
\]
so the series is normally convergent for $|z| \leq r$ because $r/r_1 \in (0, 1)$. □

A power series $F$ with $R > 0$ is called convergent. The sum of such series is a continuous function in $D(0, R)$.

6.2 The substitution map $C[[z]] \rightarrow C(R)$ is a homomorphism of rings.

We are going to prove that this homeomorphism is injective. Actually we will prove a little stronger statement.

6.3 **Theorem.** If $F \neq 0$ is a series whose radius of convergence is $R > 0$, then there exists $r \in (0, R)$, such that $F(z) \neq 0$ for $0 < |z| < r$.

**Proof.** We have $m := \text{ord } F < \infty$, so $F(z) = z^m G(z)$, where $G(0) \neq 0$. By continuity there exists an $r \in (0, R)$, such that $G(z) \neq 0$ for $|z| < r$. This proves the theorem. □

6.4 **Theorem.** If $F$ and $G$ are convergent power series, and $G(0) = 0$, then $F \circ G$ is also convergent.

**Proof.** Exercise!

6.5 **Corollary.** If $F$ is a convergent power series, and $F(0) \neq 0$ then the multiplicative inverse $F^{-1}$ is also convergent.