1. \[
\int_0^\infty \frac{\sin x}{x(x^2 + 1)} \, dx = \frac{1}{2} \text{Im p.v.} \int_{-\infty}^\infty \frac{e^{ix}}{x(x^2 + 1)} \, dx,
\]
because the integrand is even, and \(\sin x = \text{Im} \, e^{ix}\). Unlike the original integral, last integral does not exist the ordinary sense, and we use principal value.

We integrate
\[
f(z)dz = \frac{e^{iz} \, dz}{z(z^2 + 1)}
\]
over a contour which consists of an interval \((-M, M)\), a large half-circle in the upper half-plane, and a small half-circle in the upper half-plane around zero. The integral over the large half-circle tends to zero by the Jordan lemma. The integral over the small half-circle tends to \(-\pi i \text{res}_0 f\) as was computed in class.

So,
\[
\int_{-\infty}^\infty \frac{e^{iz} \, dz}{z(z^2 + 1)} = \pi i \text{res}_0 f + 2\pi i \text{res}_1 f.
\]
By computing the residues (all poles are simple) we obtain the answer:
\[
\int_0^\infty \frac{\sin x}{x(x^2 + 1)} \, dx = \frac{\pi}{2} (1 - 1/e).
\]

2. In the disc \(|z| \leq 1\) we have
\[
|z^4 - 8z| \leq |z^4| + 8|z| \leq 9 < 10 = |10|,
\]
so the number of roots is the same as that of the function 10, that is no roots.

In the disc \(|z| < 3\) we have
\[
|-8z + 10| \leq 8|z| + 10 = 34 < 81 = |z^4|,
\]
so the number of roots is the same as that of the function \(z^4\) that is 4.

Thus our function has 4 roots in the ring \(1 < |z| < 3\).

3. Answer: \(1/c_2\). This problem is completely trivial of course. Actually I wanted to ask about \(f(z) = c_2z^2 + c_3z^3\) but a misprint made it trivial.
4. a) Simple pole at zero and double poles at $2\pi k$, for every non-zero integer $k$.
   b) Essential singularity at 0.
   c) Removable singularity at zero and simple poles at $\pi k$ for every non-zero integer $k$.
   d) Removable singularity at 0 and essential singularity at $-1$.
   e) Essential singularity at 0.

5. a) $R = \infty$ by the ratio test. Of course, the root test also works but it is harder because one has to know Stirling’s formula.
   b) $R = 1/\sqrt{2}$ by the root test.
   c) $R = \pi/2$, the distance from the origin to the closest singularity.
   d) $R = 1$, by the same reason. The function is analytic in the plane with a slit along the negative ray, and cannot be analytically continued to any larger region, because it tends to infinity as $z \to -1$.
   e) $R = \infty$. The singularities at $\pm \pi$ are removable because $\sin z$ has zeros there, and these zeros cancel the zeros in the denominator. So the function is in fact analytic in the whole plane.