Hints and Solutions, HW 2

1. Let $y(t)$ be the temperature of the carcass $t$ hours after the boar was killed. According to the law of cooling

$$y' = k(40 - y),$$

where $k$ is (unknown) coefficient of proportionality. This equation is separable and the general solution is

$$y(t) = 40 + Ce^{-kt}.$$  

Using $y(0) = 100$, we obtain $C = 60$. Now the results of the ranger’s measurements give

$$40 + 60e^{-kT} = 60 \quad \text{and} \quad 40 + 60e^{-k(T+1)} = 50,$$

where $T$ is the time from the shot to the first measurement. Solving this system of equations we find that $k = \log 2$ and $T = \log 3/\log 2 \approx 1.5850$, that is one hour and thirty five minutes.

2. If at the time $t$ (in hours) we have $y(t)$ kilograms of $C$, this means that $2y(t)/3$ kg of $A$ and $y(t)/3$ kg of $B$ have been consumed, thus $10 - 2y(t)/3$ kg of $A$ and $20 - y(t)/3$ kg of $B$ remains. Thus

$$y' = k(10 - 2y/3)(20 - y/3),$$

where $k$ is a coefficient of proportionality, still unknown. The initial condition is $y(0) = 0$ (the water was pure, there was no $C$ in the beginning).

The equilibria are $y_1 = 15$ and $y_2 = 60$.

The equation is separable, and its general solution is

$$\frac{60 - y}{15 - y} = Ce^{15kt}. \tag{1}$$

Using the initial condition $y(0) = 0$, we obtain $C = 4$. To find $k$ we use the condition $y(1/3) = 6$, which gives $e^{15k} = 3/2$. Using these values of $C$ and $k$ we solve (1) and find

$$y(t) = \frac{15(1 - (2/3)^{3t})}{1 - (1/4)(2/3)^{3t}}.$$

So, for $t = 1/2$ we have $y(1/2) \approx 7.9117$.

“Eventually” the output of the substance $C$ will be 15 kg, all $A$ will be consumed, and some $B$ remain. Thus 99% of $A$ will be consumed when 99% of $C$ is made. To find when will this happen we write

$$.99 \times 15 = \frac{15(1 - (2/3)^{3t})}{1 - (1/4)(2/3)^{3t}},$$
which gives
\[ t = \frac{1}{3\log(2/3)} \log \frac{.1}{1 - .99/4} \approx 3.5521, \]
that is approximately three hours and thirty three minutes.

3. a) and c) are homogeneous; b) is exact.

5. \( b = 3. \)

6. Just differentiate (assuming that \( y \) is a function of \( x \)):
\[ e^y + xe^y y' - y^2 - 2xy y' + y' = 0, \]
or \( y'(xe^y - 2xy + 1) + e^y - y^2 = 0. \)