Homework 9

1. a) For which positive integers $m$ is it possible that $2^m$ and $2^{m+1}$ have equal sums of digits in decimal system? Hint: Every number has the same residue modulo 3, as sum of its digits in decimal system. So, for example 721, 020, 013, 214 has residue 2 modulo 3.

b) How to find out quickly, whether a given integer is divisible by 11?

2. a) Express a primitive root of 1 of degree 8 in the form $a + bi$ (your expression may contain radicals but no sines, cosines or exponentials).

b*) Do the same for a primitive root of degree 5.

3. a) A complex number $w$ is called a primitive root of 1 of degree $d$, if all roots of 1 of degree $d$ are powers of $w$. Which roots of 1 of degree 12 are primitive, and which are not?

b*) Is $\exp \frac{7\pi i}{60}$ a root of 1 of degree 120? If yes, is it primitive?

Definition. Fourier matrix of size $N \times N$ is the matrix with elements $a_{i,j} = w^{-ij}$, where $w = \exp \frac{2\pi}{N}$ and $i, j = 0, \ldots, N - 1$.

4. Write explicitly the Fourier matrix $6 \times 6$.

5. Let $A$ be the $4 \times 4$ Fourier matrix. Find $A^2$ and $A^4$.

6. a) Show that the 4-th power of every Fourier matrix is a multiple of the unit matrix.

b) How does the second power of a Fourier matrix look?

7. Find the Fourier transform of the vector $(2, 1, -2, 1)$.

8. According to Danielson and Lanczos, their first calculation at Purdue, took 10 min to find the FT of a 8-vector (all by hand, of course), 25 min for a 16-vector, 60 min for a 32-vector, and 140 min for a 64-vector. Plot these data, and discuss, whether are consistent with the theoretical result that the time is proportional to $N \log N$. Estimate the coefficient of proportionality. (Please do all calculations by hand:-)

9. Prove Parceval’s identity:

$$\sum_{n=0}^{N-1} |F(n)|^2 = N \sum_{n=0}^{N-1} |f(n)|^2.$$
10. Suppose that a vector \((a, b, c, d, e, f, g, h)\) has FT \((A, B, C, D, E, F, G, H)\). Find the vector (of dimension 4), whose FT is \((A, C, E, G)\).