Solution of the isochrone problem

Let the $x$-axis be horizontal, $h$-axis vertical, pointing down, and $x = f(h)$ be the equation of the curve. We may assume that the motion starts at the origin. Suppose that the initial speed is $v_0$ and the constant vertical component of velocity is $V$. By the Energy Conservation Law the speed $v = v(h)$ satisfies $v^2 = 2gh + v_0^2$. Denoting by $\alpha$ the angle between the tangent to the curve and vertical direction, we see that the vertical component of velocity is $v \cos \alpha$. On the other hand, $\tan \alpha = dx/dh$, Putting all this together, and using

$$
\cos^2 \alpha = (1 + \tan^2 \alpha)^{-1},
$$

we obtain the differential equation:

$$
1 + \left(\frac{dx}{dh}\right)^2 = V^2(2gh + v_0^2),
$$

This equation is separable and can be reduced to evaluation of the simple integral:

$$
x(h) = \int_0^h \sqrt{V^2(2gh + v_0) - 1} \, dy = \frac{V^2}{3g} \left( \frac{2gh}{V^2} + \frac{v_0^2}{V^2} - 1 \right)^{3/2}.
$$

It is clear from this formula (and also from the physical interpretation) that $v_0 \geq V$, that is the motion cannot start from rest. The curve we obtained is called a semi-cubic parabola.