Proof that \( t \mapsto \exp(it) : \mathbb{R} \to \mathbb{T} \) is surjective

Another proof is in Ahlfors, p. 45, or Whittaker-Watson, vol. 1, Appendix. All other authors seem to rely on the facts about trigonometric functions “proved” in high school.

A topological group \( G \) is a set with a group structure and topology, so that the group operations are continuous. This means that the maps \( G \times G \to G, (x, y) \mapsto xy \) and \( G \to G, x \mapsto 1/x \) are continuous (first of them, with respect to product topology). Morphisms of topological groups are defined as continuous homomorphisms. Examples of topological groups are \((\mathbb{Z}, +)\) with discrete topology and \((\mathbb{R}, +), (\mathbb{C}, +), (\mathbb{C}^*, \cdot), \mathbb{T}\) with their natural topologies.

**Proposition.** If \( G \) is a topological group, and \( H \subset G \) a subgroup, which contains a neighborhood of unity, then \( H \) is open and closed, that is \( H \) is a connected component of \( G \).

**Proof.** Let \( U \) be this neighborhood of the unity. If \( x \in H \) then \( xU := \{xy : y \in U\} \) is a neighborhood of \( x \), which is contained in \( H \). So \( H \) is open. Now suppose that \( x \in \overline{H} \). Then \( xU \cap H \neq \emptyset \), and we choose an element \( y \in xU \cap H \). Then \( x = yz^{-1} \) for some \( z \in U \subset H \), thus \( x \in H \). This shows that \( H \) is closed. \( \square \)

In the second lecture we defined a morphism of topological groups \((\mathbb{R}, +) \to \mathbb{T}, \ t \mapsto \exp(it)\). If \( H \subset \mathbb{T} \) is the image of this morphism, then \( H \) is a topological subgroup of \( \mathbb{T} \). We are going to prove that \( H = \mathbb{T} \).

**Lemma.** There is a neighborhood \( U \) of \( 1 \) in \( \mathbb{T} \), such that the map \( U \to \mathbb{R}, z \mapsto \Im z \), is a homeomorphism onto its image.

**Proof.** The equation of the unit circle in \( \mathbb{R}^2 \) is \( x^2 + y^2 = 1 \). For each \( |y| < 1/2 \) this equation, with respect to \( x \), has exactly one positive solution. Thus we can take \( U = \{z \in \mathbb{T} : \Re z > 0, |\Im z| < 1/2\} \). \( \square \)

Let \( V \) be the component of \( \exp^{-1}(U) \), which contains the origin. The map \( V \to \mathbb{R}, t \mapsto \Im \exp(it) = \sin t \) is differentiable with positive derivative at \( 0 \). So there exists a neighborhood \( V' \) of \( 0 \) in \( \mathbb{R} \), such that the restriction \( \sin : V' \to \mathbb{R} \) is a homeomorphism onto its image. It follows that \( t \mapsto \exp(it) : V' \to \mathbb{T} \) is a homeomorphism onto the image, thus the image \( U' \) is a neighborhood of \( 1 \) in \( \mathbb{T} \).

We have seen, that the subgroup \( H \subset \mathbb{T} \) contains a neighborhood of the unity, namely \( U' \). Thus by the Proposition \( H = \mathbb{T} \), because \( \mathbb{T} \) is connected.