CHAPTER 10

Limits of Functions

In Chapter 4 we described the limit of a sequence in terms of approximations. We may describe limits of functions similarly.

Example 1. The Acme Geometry Company, manufacturers of precision geometric figures, has received an order for a batch of squares, each with area 9 square inches, \(\pm 10^{-6}\). How accurately must the sides of the squares be cut?

Solution: Let \(s\) be the side length of our square. To be within tolerance, we require

\[9 - 10^{-6} < s^2 < 9 + 10^{-6}\]

Taking square roots:

\[2.999999833 \ldots < s < 3.000000167 \ldots\]

This will certainly be true if

\[2.9999999 < s < 3.0000001\]

Thus, it suffices to make our cut accurate to within \(\pm 10^{-7}\).

It is rare that we can explicitly solve the inequalities as was done in Example 1. The following example shows how to deal with more complicated cases.

Example 2. We wish to make a rectangular box with volume \(6 \pm .1\) cubic units by cutting a square of side length 2 from each corner of a \(7 \times 5\) piece of paper and bending up the sides. How accurately must the sides of the squares be cut? (Assume that all 4 of the cut out squares are identical.)

Solution: According to Figure 1, the volume \(V\) of the box is

\[V(x) = (5 - 2x)(7 - 2x)x = 4x^3 - 24x^2 + 35x\]
where $x$ is the side length of each square. We need to find a number $\delta > 0$ such that
\begin{equation}
|V(x) - 6| < .1
\end{equation}
whenever $x = 2 \pm \delta$. We note that
\begin{align*}
V(x) - 6 &= 4x^3 - 24x^2 + 35x - 6 \\
&= (x - 2)(4x^2 - 16x + 3)
\end{align*}

We refer to the $(x - 2)$ factor as the “gold” since it is small for $x$ near 2. We refer to the $(4x^2 - 16x + 3)$ factor as the “trash” since it does not help in making $|V(x) - 6|$ small. Our first step will be to find a specific number $B$ such that for all $x$ sufficiently close to 2,
\begin{equation*}
|4x^2 - 16x + 3| \leq B.
\end{equation*}
(We refer to this as “bounding the trash.”)

Specifically, suppose that $x = 2 \pm 1$ i.e. $1 < x < 3$. Then
\begin{align*}
|4x^2 - 16x + 3| &\leq 4|x|^2 + 16|x| + 3 \\
&< 4 \cdot 9 + 16 \cdot 3 + 3 = 87
\end{align*}

Hence, for $x = 2 \pm 1$,
\begin{equation*}
|V(x) - 6| = |x - 2| |4x^2 - 16x + 3| < 87|x - 2|
\end{equation*}
Hence, inequality (2) holds if
\begin{align*}
87|x - 2| &< 10^{-1} \\
|x - 2| &< \frac{10^{-1}}{87}
\end{align*}
Hence we can set \( \delta \) to be the smaller of 1 (to guarantee that \( x = 2 \pm 1 \)) and \( \frac{10^{-1}}{87} \), i.e. \( \delta = \frac{10^{-1}}{87} \).

**Remark:** Examples 1 and 2 demonstrate a type of approximation problem in which we are given (1) a function \( y = f(x) \), (2) a number \( L \) (the target value), (3) a number \( \epsilon > 0 \) (the tolerance) and (4) a value \( a \) on the \( x \)-axis. Our goal is to find a number \( \delta > 0 \) (the confidence value) such that

\[
f(x) = L \pm \epsilon
\]

whenever \( x = a \pm \delta \). In Example 1 our target value was 9, our tolerance was \( \epsilon = 10^{-6} \) and our confidence value was \( \delta = 10^{-7} \). In Example 2 our target value was 6, our tolerance was \( \epsilon = .1 \) and our confidence value was \( \delta = .1/87 \).

The solution to Example 2 demonstrates one technique for solving approximation problems:

1. Choose an initial value \( \delta_o \) for \( \delta \) and assume initially that \( x = a \pm \delta_o \), i.e. \( a - \delta_o < x < a + \delta_o \).
2. By factoring, write

\[
f(x) - L = (x - a)g(x)
\]

where \( g(x) \) is some function. We refer to the \( (x - a) \) term as the “gold” because it is small when \( x \) is near \( a \). The \( g(x) \) term is referred to as “trash.”
3. Bound the trash over the interval \( (a - \delta_o, a + \delta_o) \), i.e. find a number \( B \) such that

\[
|g(x)| \leq B
\]

for all \( x \in (a - \delta_o, a + \delta_o) \).
4. It then follows that

\[
|f(x) - L| = |x - a||g(x)| \leq |x - a|B
\]

for all \( x \in (a - \delta_o, a + \delta_o) \). This will be \( < \epsilon \) if \( |x - a| < \epsilon/M \). Thus, we can choose \( \delta \) to be the smaller of \( \delta_o \) and \( \epsilon/M \).

There is also a geometric way of solving Example 2:

**Graphical Solution of Example 2:** Using a graphing calculator, we plot \( V \) using the viewing window defined by \([5.9, 6.1]\) on the \( y \)-axis.
and some small interval on the $x$-axis containing $x = 2$, say $[1.8, 2.2]$. (Figure 2.)

**Figure 2. Graphical Solution of Example 2**

Any point on the visible portion of the graph has its $V$ coordinate within tolerance. Using the trace feature on the calculator, we find that the $x$-values for this piece of the graph range from $x = 1.993$ to $x = 2.005$. Any $x$ within $\pm .005$ of 2 will lie within this range. Thus, *if our graph is correct*, we can choose $\delta = .005$.

As a check, we compute that $V(1.995) = 6.065$ and $V(2.005) = 5.94$, both of which are within $\pm .1$ of 6. Our graph indicates that $V$ is decreasing over $[1.995, 2.005]$. Granted this, it follows that for all $x$ in this interval

$$6.065 \geq V(x) \geq 5.94$$

proving that $V(x)$ is within $\pm .1$ of 6. Hence our value of $\delta$ is correct.

**Remark:** The analytical solution suggests that we need to cut our sides to a tolerance of $\frac{.1}{87} \approx .001$ units. However, the graphical solution shows that in fact we only need $\pm .005$ accuracy. Thus, the graphical solution is "better." However, the analytical solution has the advantage that it yields a formula for $\delta$ as a function of $\epsilon$. Specifically, the work done in Example 2 showed that for $0 < x < 3$,

$$|V(x) - 6| < \frac{87}{|x - 2|}.$$ 

Hence, $|V(x) - 6| < \epsilon$ will be true if $|x - 2| < \epsilon/87$ showing that we can set $\delta$ to be the smaller of 1 and $\epsilon/87$. 
To relate Examples 1 and 2 to the concept of limit, consider what we mean by the statement
\[ \lim_{x \to a} f(x) = L. \]
One way of describing this might be “\( f(x) \) becomes close to \( L \) as \( x \) becomes close to, but not equal to, \( a \).” (We exclude \( x = a \) since we are describing what happens as \( x \) “approaches” \( a \).) In terms of approximations we could say that \( f(x) \) can be made to approximate \( L \) as closely as desired by making \( x \) sufficiently close to \( a \), without necessarily equaling \( a \). In more precise terms, we adopt the following definition:

**Definition 1.** We say that
\[ \lim_{x \to a} f(x) = L \]
provided that for all numbers \( \epsilon > 0 \) there is a number \( \delta > 0 \) such that
\[ |f(x) - L| < \epsilon \]
for all \( x \) satisfying \( 0 < |x - a| < \delta \).

This definition can be thought of in terms of a factory. The boss says, “I want \( f(x) \) to approximate \( L \) to within \( \pm 1 \). How close to \( a \) must \( x \) be?”

You respond, “Were O.K. if \( x \) is within \( \pm .01 \) of \( a \).”

The boss says, “I changed my mind. Now I want \( f(x) \) to approximate \( L \) to within \( \pm .001 \).”

You respond, “We can do it, but now we’ll need \( x \) to be within \( \pm .00005 \) of \( a \).”

If you can always satisfy the boss, no matter how many times he changes his mind, then \( \lim_{x \to a} f(x) = L \). Proving that you can always find \( \delta \) usually requires producing a formula for \( \delta \) as a function of \( \epsilon \).

**Remark:** The steps in doing the typical limit proof are:

1. Assume that a value of \( \epsilon \) is given.
2. State an appropriate \( \delta \). (It will depend on \( \epsilon \).)
3. Prove that the stated value of \( \delta \) works.

**Example 3.** Use the definition of limit to prove that
\[ \lim_{x \to 1} \frac{x}{2x + 1} = \frac{1}{3} \]
Solution:

**Scratch Work:** Let $\epsilon > 0$ be given. We want

\[
\left| \frac{x}{2x+1} - \frac{1}{3} \right| < \epsilon
\]

\[
\left| \frac{x-1}{3(2x+1)} \right| = \left| x-1 \right| \left| \frac{1}{3(2x+1)} \right| < \epsilon
\]

*The term $1/3(2x + 1)$ is the “trash.” To bound it assume that $\delta_o = 1$ so that $|x-1| < \delta_o$ implies $x \in [0, 2]$. Then $3(2x+1) \geq 3$ and

\[
|x-1| \left| \frac{1}{3(2x+1)} \right| < \frac{1}{3} |x-1|
\]

This will be $< \epsilon$ if $|x-1| < 3\epsilon$. Thus, we choose $\delta$ to be the smaller of $1$ and $3\epsilon$.

**Our Proof:** Let $\epsilon > 0$ be given and let $\delta = \min\{1, 3\epsilon\}$. Assume that $|x-1| < \delta$. Then $x \in [0, 2]$.

Hence

\[
\left| \frac{x}{2x+1} - \frac{1}{3} \right| = \left| \frac{x-1}{3(2x+1)} \right|
\]

\[
= \left| x-1 \right| \left| \frac{1}{3(2x+1)} \right| < |x-1| \frac{1}{3} \quad \text{(From the scratch work.)}
\]

\[
< \frac{\delta}{3} < \frac{3\epsilon}{3} = \epsilon
\]

Hence, $|x-1| < \delta$ implies $|f(x) - \frac{1}{3}| < \epsilon$, proving the limit statement.

**Example 4.** Use the definition of limit to prove that

\[
\lim_{x \to 3} \sqrt{7} - 2x = 1
\]

**Scratch Work:** Let $\epsilon > 0$ be given. We want

\[
|\sqrt{7} - 2x - 1| < \epsilon
\]
Rationalizing:

\[
\sqrt{7 - 2x} - 1 = \frac{(\sqrt{7 - 2x} - 1)(\sqrt{7 - 2x} + 1)}{\sqrt{7 - 2x} + 1}
\]

\[
= \frac{(7 - 2x) - 1}{\sqrt{7 - 2x} + 1}
\]

\[
= \frac{-2(x - 3)}{\sqrt{7 - 2x} + 1} = (x - 3) \frac{-2}{\sqrt{7 - 2x} + 1}
\]

Thus

\[
|x - 3| \left| \frac{-2}{\sqrt{7 - 2x} + 1} \right|
\]

\[
= |x - 3| \frac{2}{\sqrt{7 - 2x} + 1} \leq \frac{2}{1} |x - 3|
\]

This is \( < \epsilon \) if \( |x - 3| < \epsilon/2 \). Hence we set \( \delta = \epsilon/2 \). 

There is, however, a small problem. The square root is only defined if

\[
7 - 2x \geq 0
\]

\[
7 \geq 2x
\]

\[
3.5 \geq x
\]

This is true if \( x = 3 \pm .5 \). Hence we also need to choose \( \delta < .5 \).

Our Proof: Let \( \epsilon > 0 \) be given and let \( \delta = \min\{.5, \epsilon/2\} \). Assume that \( |x - 1| < \delta \). Then \( x \in [2.5, 3.5] \). For such \( x \)

\[
|\sqrt{7 - 2x} - 1| = |x - 3| \frac{2}{\sqrt{7 - 2x} + 1}
\]

\[
\leq 2|x - 3| < 2\delta < 2\frac{\epsilon}{2} = \epsilon
\]

Hence, \( |x - 3| < \delta \) implies \( |f(x) - 1| < \epsilon \), proving the limit statement.

What’s the point?

In the past, students have made comments such as, “I can do this \( \delta\)-\( \epsilon \) stuff, but I really don’t see any value to it. It doesn’t help me compute the limit—I need to know the limit before I start. In fact, in most of the exercises, I can find the limit by plugging \( a \) into the formula.”
The simplest answer is that the $\delta$-$\varepsilon$ technique is a way of **proving** that the limit is what you think it should be. We guess that

$$\lim_{x \to 1} \frac{x}{2x + 1} = \frac{1}{3}$$

The work in Example 3 proves it.

The skeptical student might respond, “I don’t have to guess. As $x \to 1$, the numerator gets close to 1 and the denominator gets close to 3 so the ratio must be getting close to 1/3. Isn’t this proof enough?”

The answer is, “Yes and no.” The term “close” is much too imprecise to be in a proof. However, here is a valid version of the student’s proof:

$$\lim_{x \to 1} \frac{x}{2x + 1} = \lim_{x \to 1} \frac{x}{\lim_{x \to 1}(2x + 1)}$$

$$= \frac{1}{\lim_{x \to 1}(2x) + \lim_{x \to 1}1}$$

$$= \frac{1}{2(\lim_{x \to 1} x) + 1} = \frac{1}{2 + 1} = \frac{1}{3}$$

This argument relies on the Sum, Product, and Quotient Theorems. Once these theorems have been proved, the above argument just as valid as the $\delta$-$\varepsilon$ argument from Example 3. However, proving the Sum, Product, and Quotient Theorems requires $\delta$-$\varepsilon$ arguments. Many, many, other theorems in mathematics are proved using $\delta$ – $\varepsilon$ methods. The purpose of using $\delta$-$\varepsilon$ arguments to prove “obvious” statements such as those in Examples 3 and 4 above is to get a better understanding of the $\delta$-$\varepsilon$ technique so as to better understand both what a limit really is and to better understand the proofs of general facts such as the theorems form this section. In fact, the $\delta$-$\varepsilon$ idea is essential for proving most of the theorems found in any calculus text. It is, in a sense, the corner stone of calculus.

**Theorem 1** (Sum Theorem). *Suppose that $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ both exist. Then*

$$\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$
Proof Let \( L = \lim_{x \to a} f(x) \) and \( M = \lim_{x \to a} g(x) \). Let \( \epsilon > 0 \) be given. There are numbers \( \delta_1 \) and \( \delta_2 \) such that
\[
|f(x) - L| < \frac{\epsilon}{2}
\]
whenever \( 0 < |x - a| < \delta_1 \) and
\[
|g(x) - M| < \frac{\epsilon}{2}
\]
whenever \( 0 < |x - a| < \delta_2 \). Let \( \delta \) be smaller than both \( \delta_1 \) and \( \delta_2 \). Then, for \( 0 < |x - a| < \delta \),
\[
|f(x) + g(x) - (L + M)| = |(f(x) - L) + (g(x) - M)|
\]
\[
< |f(x) - L| + |g(x) - M| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\]
which proves the theorem.

In preparation for the proof of the product theorem, we prove the following special case:

**Lemma 1.** Suppose that \( \lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x) \). Then \( \lim_{x \to a} f(x)g(x) = 0 \).

*Proof* Let \( \epsilon > 0 \) be given. We need to show that there is a \( \delta > 0 \) such that
\[
|f(x)g(x) - 0| = |f(x)g(x)|
\]
\[
= |f(x)||g(x)| < \epsilon
\]
for all \( 0 < |x - a| < \delta \).

This will be true if both of the following inequalities hold for all \( 0 < |x - a| < \delta \):
\[
|f(x)| < \sqrt{\epsilon}
\]
\[
|f(x)| < \sqrt{\epsilon}
\]

However, since \( \lim_{x \to a} f(x) = 0 \), we know that there is a number \( \delta_1 \) such that the first inequality in (5) holds for all \( 0 < |x - a| < \delta_1 \). Similarly, there is a number \( \delta_2 \) such that the second inequality in (5) holds for all \( 0 < |x - a| < \delta_2 \). Both inequalities hold for \( 0 < |x - a| < \delta \) where \( \delta = \min\{\delta_1, \delta_2\} \). Hence, inequality (4) holds for all \( 0 < |x - a| < \delta \), proving our lemma.

We also note the following simple results which are left as exercises. (Exercises 12,11, and 10 beginning on page 156.)
Lemma 2. If $C$ is any constant, then $\lim_{x \to a} C = C$.

Lemma 3. If $C$ is any constant, then $\lim_{x \to a} Cf(x) = C \lim_{x \to a} f(x)$.

Lemma 4. If $f(x)$ is a function, then $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a}(f(x) - L) = 0$.

We are now ready to prove the product theorem.

Theorem 2 (Product Theorem). Suppose that $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ both exist. Then
\[ \lim_{x \to a} (f(x)g(x)) = (\lim_{x \to a} f(x))(\lim_{x \to a} g(x)) \]

Proof. Let $L = \lim_{x \to a} f(x)$ and $M = \lim_{x \to a} g(x)$. Our theorem is equivalent with the statement
\[ \lim_{x \to a} (f(x)g(x)) = LM \]

However, from Lemma 4,
\[ \lim_{x \to a} (f(x) - L) = 0 \]
\[ \lim_{x \to a} (g(x) - M) = 0 \]

Hence, from Lemma 1,
\[ \lim_{x \to a} (f(x) - L)(g(x) - M) = 0 \]

On the other hand
\[ (f(x) - L)(g(x) - M) = f(x)g(x) - Mf(x) - Lg(x) + LM. \]

Solving for $f(x)g(x)$, we get
\[ f(x)g(x) = (f(x) - L)(g(x) - M) + Mf(x) + Lg(x) - LM. \]

Hence, using equation (6), the Sum Theorem, Lemma 2, and Lemma 3, we see
\[ \lim_{x \to a} (f(x)g(x)) = 0 + M \lim_{x \to a} f(x) + L \lim_{x \to a} g(x) - LM \]
\[ = ML + LM - LM = LM \]

proving the Product Theorem. \qed

Theorem 3 (Inverse). Suppose that $\lim_{x \to a} f(x) \neq 0$. Then
\[ \lim_{x \to a} \frac{1}{f(x)} = \frac{1}{\lim_{x \to a} f(x)} \]
Case 1: \( \lim_{x \to a} f(x) = 1 \).

We note that

\[
\left| \frac{1}{f(x)} - 1 \right| = \left| \frac{1 - f(x)}{|f(x)|} \right| = \frac{|f(x) - 1|}{|f(x)|}
\]

Since \( \lim_{x \to a} f(x) = 1 \), for all \( \epsilon > 0 \), there is a \( \delta > 0 \) such that

\[
|f(x) - 1| < \epsilon
\]

for \( 0 < |x - a| < \delta \).

Letting \( \epsilon = .5 \) in (7) we see that there is a \( \delta_1 > 0 \) such that

\[-.5 < f(x) - 1 < .5 \]
\[.5 < f(x) < 1.5\]

for \( 0 < |x - a| < \delta_1 \). Hence, for such \( x \),

\[
\left| \frac{f(x) - 1}{f(x)} \right| < \frac{|f(x) - 1|}{.5} = 2|f(x) - 1|
\]

Now, let \( \epsilon > 0 \) be given. Replacing \( \epsilon \) with \( \epsilon/2 \) in (7) shows that there is a \( \delta_2 > 0 \) such that

\[
|f(x) - 1| < \epsilon/2
\]

for \( 0 < |x - a| < \delta_2 \). Let \( \delta \) be smaller than both \( \delta_1 \) and \( \delta_2 \). Then

\[
\left| \frac{1}{f(x)} - 1 \right| < 2|f(x) - 1| < 2 \frac{\epsilon}{2} = \epsilon
\]

for \( 0 < |x - a| < \delta \). This finishes Case 1.

General Case:

Suppose that \( \lim_{x \to a} f(x) = L \neq 0 \). Then from Lemma 3

\[
\lim_{x \to a} \frac{f(x)}{L} = 1
\]

Hence, from Case 1

\[
1 = \lim_{x \to a} \frac{L}{f(x)} = L \lim_{x \to a} \frac{1}{f(x)}
\]

Hence

\[
\lim_{x \to a} \frac{1}{f(x)} = \frac{1}{L}
\]

as desired.

The following follows from the Product Theorem and Theorem 3 and is left as an exercise.
THEOREM 4 (Quotient Theorem). Suppose that \( \lim_{x \to a} f(x) \) and \( \lim_{x \to a} g(x) \) both exist and \( \lim_{x \to a} g(x) \neq 0 \). Then
\[
\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}
\]

The following example is similar to many of the exercises in the notes.

EXAMPLE 5. Suppose that \( \lim_{x \to a} f(x) = 3 \). Prove, using \( \delta \) and \( \epsilon \), that
\[
\lim_{x \to a} \frac{1}{f(x) - 5} = -\frac{1}{2}
\]

Scratch work: Let \( \epsilon > 0 \) be given. We want
\[
\left| \frac{1}{f(x) - 5} + \frac{1}{2} \right| < \epsilon
\]
(8)
\[
\frac{|3 - f(x)|}{2|f(x) - 5|} < \epsilon
\]
\[
|f(x) - 3| \frac{1}{2|f(x) - 5|} < \epsilon
\]

The term on the left is our “gold” since it becomes small as \( x \) approaches \( a \). The other term is our “trash” which we will bound. Specifically, we reason that for all \( x \) sufficiently close to \( a \), \( f(x) = 3 \pm 1 \). Thus, for such \( x \),
\[
2 < f(x) < 4
\]
\[
-3 < f(x) - 5 < -1
\]
\[
1 < |f(x) - 5| < 3
\]
Hence
(9)
\[
|f(x) - 3| \frac{1}{2|f(x) - 5|} < \frac{1}{2} |f(x) - 3|
\]
This is \( < \epsilon \) if \( |f(x) - 3| < 2\epsilon \), which is true for all \( x \) sufficiently close to \( a \).

Proof: We know that for all \( \epsilon > 0 \), there is a \( \delta > 0 \) such that
\[
|f(x) - 3| < \epsilon \text{ for } 0 < |x - a| < \delta
\]
Specifically, for \( \epsilon = 1 \), we see that
there is a $\delta_o$ such that

$$
\begin{align*}
|f(x) - 3| < 1 \\
-1 < f(x) - 3 < 1 \\
-3 < f(x) - 5 < -1 \\
1 < |f(x) - 5| < 3
\end{align*}
$$

(10)

Now let $\epsilon > 0$ be given and choose $\delta_1 > 0$ so that

$$
|f(x) - 3| < 2\epsilon
$$

for $0 < |x - a| < \delta_1$. Let $\delta = \min\{\delta_1, \delta_o\}$. Then both (10) and (11) hold for $0 < |x - a| < \delta$. Hence, from (8), $0 < |x - a| < \delta$ implies that

$$
\left| \frac{1}{f(x) - 5} + \frac{1}{2} \right| < \epsilon
$$

proving the limit statement.

**IMPORTANT!**: Unlike the limit problems done earlier, we never stated an explicit value for $\delta$ in Example 5. All we said was choose $\delta$ so that (10) and (11) both hold for $0 < |x - a| < \delta$. The value $2\epsilon$ in (11) is not $\delta$. We would need to solve (11) for $x$ to find $\delta$.

![Figure 3. The “delta machine”](image)
solution to Example 5 says that we need to turn the $|x - a|$ knob
until the value of $|f(x) - L|$ is below both $2\epsilon$ and 1. The value the
$|x - a|$ knob points to is $\delta$. We know that there is such a $\delta$ since the
definition of $\lim_{x \to a} f(x) = L$ tells us, in effect, that we can make the
$|f(x) - L|$ meter point as close to 0 as we wish by turning the $|x - a|$ knob sufficiently far.

Exercises

(1) We wish to produce a cube with volume $27 \pm .01$. Find a
value of $\delta$ such any cube with side length $3 \pm \delta$ will have
volume in the stated interval.

(2) A farmer bought exactly 10 feet of fencing wire to fence in a
$2 \times 6$ plot of land beside his barn. (Figure 3). How accurately
must he cut the short side of his fence if he wants the area
to be $12 \pm .1$ square feet. (Why he wants this is beyond
me!) Solve both graphically and analytically. \textit{Hint:} Call the
actual length of the short side $x$.

![Figure 3](image)

(3) For each of the following functions:
   \begin{enumerate}
   \item Find a value of $\delta$ for which $f(x)$ approximates $f(a)$ to
   within $\pm \epsilon$ for all $x$ within $\pm \delta$ of $a$ where $a$ and $\delta$
   are as stated. Solve \textit{graphically}.
   \item Re-do (a) \textit{analytically}.
   \item Give a general formula for $\delta$ in terms of $\epsilon$.
   \end{enumerate}
(i) \( f(x) = 5x + 3 \quad a = 2 \quad \epsilon = .003 \)
(ii) \( f(x) = 3x - 1 \quad a = -1 \quad \epsilon = 10^{-2} \)
(iii) \( f(x) = x^2 - 4x + 5 \quad a = 3 \quad \epsilon = .02 \)
(iv) \( f(x) = x^3 - 7x^2 + 3 \quad a = -1 \quad \epsilon = .01 \)
(v) \( f(x) = \frac{1}{2x + 3} \quad a = 3 \quad \epsilon = .025 \)
(vi) \( f(x) = \frac{1}{1 - 2x} \quad a = 0 \quad \epsilon = .04 \)
(vii) \( f(x) = \sqrt{3 + 4x} \quad a = 1 \quad \epsilon = .01 \)
(viii) \( f(x) = \frac{1}{\sqrt{5 + 2x}} \quad a = 2 \quad \epsilon = .05 \)

(4) Find a value of \( \delta \) such that \( \frac{\sin x}{x} \) approximates 1 to within ±0.001 for all \( x \) within ±\( \delta \) of 0, \( x \neq 0 \). Repeat for \( \epsilon = .0005 \).

Solve this problem graphically—not analytically.

(5) Obtain evidence for the following limit statement by finding an appropriate \( \delta \) for (i) \( \epsilon = .5 \), (ii) \( \epsilon = .1 \), (iii) \( \epsilon = .01 \).
Reason Graphically.

\[
\lim_{x \to 0} \frac{\sin x}{x} = 1
\]
(6) Find the following limits and use the definition of limit to prove your answer.

(a) \( \lim_{x \to 2} (5x + 3) \)
(b) \( \lim_{x \to 3} 7 \)
(c) \( \lim_{x \to -1} \frac{1}{x^2} \)
(d) \( \lim_{x \to -1} \frac{1}{x^2} \)
(e) \( \lim_{x \to -0.05} \frac{1}{x^3} \)
(f) \( \lim_{x \to -1} (3x - 1) \)
(g) \( \lim_{x \to 3} (x^2 - 4x + 5) \)
(h) \( \lim_{x \to -1} (x^3 - 7x^2 + 3) \)
(i) \( \lim_{x \to 3} \frac{1}{2x + 3} \)
(j) \( \lim_{x \to 0} \frac{1}{1 - 2x} \)
(k) \( \lim_{x \to 1} \sqrt{3 + 4x} \)
(l) \( \lim_{x \to 2} \frac{1}{\sqrt{5 + 2x}} \)

(7) Evaluate each of the limits (a)-(d) using the limit theorems from the notes, together with the following results:

(a) For \( n \in \mathbb{N} \), \( \lim_{x \to a} x^n = a^n \). (Power Theorem)
(b) If \( \lim_{x \to a} f(x) > 0 \), then \( \lim_{x \to a} \sqrt{f(x)} = \sqrt{\lim_{x \to a} f(x)} \). (Square Root Theorem)

Do your proof in a step-by-step manner, stating the main theorems being used. We have done (a) as an example.
Solution of (a):

\[
\lim_{x \to 2} \frac{3x^2 - \sqrt{x + 2}}{x^2 + 1}
\]

\[
= \lim_{x \to 2} \frac{(3x^2 + (-1)\sqrt{x + 2})}{(x^2 + 1)}
\]

Quotient Theorem

\[
= \lim_{x \to 2} (3x^2) + \lim_{x \to 2} (-1)(\sqrt{x + 2})
\]

Sum Theorem

\[
= \frac{3 \lim_{x \to 2} (x^2) + (-1)\sqrt{\lim_{x \to 2} (x^2 + 2)}}{4 + 1}
\]

Constant, Power, and Root Theorems

\[
= \frac{3 \cdot 4 + (-1)\sqrt{2 + 2}}{5} = 2
\]

Power and Sum Theorems

(a) \( \lim_{x \to 2} \frac{3x^2 - \sqrt{x + 2}}{x^2 + 1} \)

(b) \( \lim_{x \to 1} \frac{x^2 + 2x + 1}{x^3 + 1} \)

(c) \( \lim_{x \to 3} \left( x^2 + \frac{x}{x^2 + 1} \right) (x^3 - 7x + 2) \)

(d) \( \lim_{x \to -1} \sqrt{(x^2 + 5)(x^2 + 2)} \)

(8) Prove the following:

(a) \( \lim_{x \to a} x = a \). (Use a \( \delta-\epsilon \) argument.)

(b) \( \lim_{x \to a} x^2 = a^2 \). (Use the Product Theorem and \( x^2 = x \cdot x \).

(c) \( \lim_{x \to a} x^3 = a^3 \). (Use the Product Theorem, part (b), and \( x^3 = x \cdot (x^2) \).

(d) \( \lim_{x \to a} x^4 = a^4 \). (Use the Product Theorem, part (c), and \( x^4 = x \cdot (x^3) \).

(e) Suppose that \( \lim_{x \to a} x^n = a^n \) has been proven for some \( n \). Use this and the Product Theorem to show that \( \lim_{x \to a} x^{n+1} = a^{n+1} \). It follows that \( \lim_{x \to a} x^n = a^n \) for all \( n \).
(9) Suppose that \( \lim_{x \to a} f(x) = 3 \). Use a \( \delta-\epsilon \) argument to prove:

(a) \( \lim_{x \to a} \frac{3}{f(x) + 1} = \frac{3}{4} \)

(b) \( \lim_{x \to a} \sqrt{f(x)} + 1 = 2 \)  
   \( \text{Hint: Rationalize} \)

(c) \( \lim_{x \to a} \frac{3}{\sqrt{f(x)} + 1} = \frac{3}{2} \)  
   \( \text{Hint: Rationalize} \)

(d) \( \lim_{x \to a} f(x)^2 = 9 \)

(e) \( \lim_{x \to a} f(x)^3 = 27 \)

(f) \( \lim_{x \to a} \frac{f(x)}{f(x) + 1} = \frac{3}{4} \)

(10) Use a \( \delta-\epsilon \) argument to prove Lemma 2.

(11) Use a \( \delta-\epsilon \) argument to prove Lemma 3.

(12) Use a \( \delta-\epsilon \) argument to prove Lemma 4.

(13) Suppose that \( \lim_{x \to a} f(x) \) and \( \lim_{x \to a} g(x) \) both exist. Use a
   \( \delta-\epsilon \) argument to prove that
   \[
   \lim_{x \to a} (f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)
   \]
   Your proof will be very similar to that of the Sum Theorem.

(14) Suppose that \( \lim_{x \to a} f(x) \) and \( \lim_{x \to a} g(x) \) both exist. Use a
   the Sum Theorem and the Constant Theorem to prove that
   \[
   \lim_{x \to a} (f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)
   \]

(15) Suppose that \( \lim_{x \to a} f(x) \), \( \lim_{x \to a} g(x) \), and \( \lim_{x \to a} h(x) \) all
   exist. Use a \( \delta-\epsilon \) argument to prove that
   \[
   \lim_{x \to a} (f(x) + g(x) + h(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) + \lim_{x \to a} h(x)
   \]
   Your proof will be very similar to that of the Sum Theorem.

(16) Suppose that for all \( x \), \( f(x) < g(x) < h(x) \) and \( \lim_{x \to a} f(x) = L = \lim_{x \to a} h(x) \). Use a \( \delta-\epsilon \) argument to prove that \( \lim_{x \to a} g(x) = L \).  
   \( \text{Hint: Write } |g(x) - L| < \epsilon \text{ in the equivalent form } L - \epsilon < g(x) < L + \epsilon. \)  
   Similarly for \( f(x) \) and \( h(x) \).

(17) Suppose that \( \lim_{x \to a} f(x) = L \) where \( L > 0 \). Prove that there is a \( \delta > 0 \) such that \( f(x) > 0 \) for all \( 0 < |x - a| < \delta \).

(18) Suppose that \( \lim_{x \to a} f(x) = L \) where \( L < 0 \). Prove that there is a \( \delta > 0 \) such that \( f(x) < 0 \) for all \( 0 < |x - a| < \delta \).
(19) Suppose that \(x_n\) is a sequence such that \(\lim_{n \to \infty} x_n = a\) and \(x_n \neq a\) for all \(n\). Suppose also that \(\lim_{x \to a} f(x) = L\). Prove that \(\lim_{n \to \infty} f(x_n) = L\).

(20) Let \(f(x) = \sin(1/x)\).
(a) Find a sequence \(x_n\), \(\lim_{n \to \infty} x_n = 0\), such that \(f(x_n) = 1\) for all \(n\).
(b) Find a sequence \(x_n\), \(\lim_{n \to \infty} x_n = 0\), such that \(f(x_n) = -1\) for all \(n\).
(c) Explain how it now follows from the result of Exercise 19 that \(\lim_{x \to 0} f(x)\) does not exist.

(21) Repeat Exercise 20 for (a) \(f(x) = \cos(1/x)\), (b) \(f(x) = \sin(1/x^2)\) (c) \(f(x) = \cos(1/x^2)\).

(22) Use a \(\delta\)-\(\epsilon\) argument to prove the following formulas. Note: If \(|x - a| < 1\), then \(|x| = |(x - a) + a| < |x - a| + |a| < 1 + |a|\).

(a) \(\lim_{x \to a} x^2 = a^2\)
(b) \(\lim_{x \to a} x^3 = a^3\)
(c) \(\lim_{x \to a} x^4 = a^4\)
(d) \(\lim_{x \to a} x^n = a^n\) \(n \in \mathbb{N}\)
(e) \(\lim_{x \to 1} \frac{1}{x} = 10\)
(f) \(\lim_{x \to a} \frac{1}{x} = \frac{1}{a} \quad a \neq 0\) \(\text{Hint: Assume } \delta_o = a \pm |a|/2\)
(g) \(\lim_{x \to 0^+} \frac{1}{x^2} = 10000\)
(h) \(\lim_{x \to a} \frac{1}{x^2} = \frac{1}{a^2} \quad a \neq 0\)
(i) \(\lim_{x \to a} \frac{1}{x^3} = \frac{1}{a^3} \quad a \neq 0\)
(j) \(\lim_{x \to a} Cx + D = Ca + D \quad C, D \in \mathbb{R}\)
(k) \(\lim_{x \to a} Cx^2 + D = Ca^2 + D \quad C, D \in \mathbb{R}\)

(23) Suppose that \(\lim_{x \to a} f(x) = L\) where \(L > 0\). Use a \(\delta\)-\(\epsilon\) argument to prove that \(\lim_{x \to a} \sqrt{f(x)} = \sqrt{L}\). \(\text{Hint: Rationalize.}\)
(24) Suppose that \( \lim_{x \to a} f(x) = L \). Use a \( \delta-\epsilon \) argument to prove that \( \lim_{x \to a} (f(x))^2 = L^2 \).

(25) Suppose that \( \lim_{x \to a} f(x) = L \). Use a \( \delta-\epsilon \) argument to prove that \( \lim_{x \to a} (f(x))^3 = L^3 \).

(26) (a) Find an example of functions \( f(x) \) and \( g(x) \) such that \( \lim_{x \to 0} f(x) = 0 \) but \( \lim_{x \to 0} f(x)g(x) = 1 \).

(b) Suppose that there is a number \( M \) such that \( |g(x)| < M \) for all \( x \). Use a \( \delta-\epsilon \) argument to prove that if \( \lim_{x \to a} f(x) = 0 \) then \( \lim_{x \to a} f(x)g(x) = 0 \).

(27) Suppose that \( \lim_{x \to a} g(x) \) exists. Show that there is a \( \delta > 0 \) and a constant \( M \) such that \( |g(x)| \) for \( 0 < |x-a| < \delta \).

*Hint:* \( |g(x)| = |(g(x) - L) + L| \).