CHAPTER 3

Rates of Growth

Inequalities are very useful in comparing rates of growth of functions.

**Definition 1.** If $f$ and $g$ are two functions, we say that $f$ grows faster than $g$ if there is a value $N$ such that $f(x) > g(x)$ for all $x > N$.

(See Figure 1.)

![Figure 1. $f$ grows faster than $g$](image)

The function $f(x) = Cx^a$ where $C$ and $a$ are both positive, exhibits **power growth** in that it grows like a power of $x$. The following example illustrates that the larger the power, the faster the growth, regardless of the value of $C$.

**Example 1.** Prove that $(.01)x^4$ grows faster than $x^2$.

**Scratch work:** According to Definition 1, we need to find a value $N$ such that $(.01)x^4 > x^2$.
for all $x > N$. Division by $(.01)x^2$ followed by taking the square root produces

$$x^2 > 100$$

$$x > 10$$

Thus, we can take $N = 10$.

For our formal solution, we state the value of $N$ and reverse the above steps to prove that the stated value works.

**Solution:** Let $N = 10$. Assume $10 > x$. We may square this inequality because $y = x^2$ is an increasing function for $x > 0$, obtaining

$$x^2 > 100$$

We may multiply by $(.01)x^2$ since this term is positive, showing that

$$(.01)x^4 > x^2$$

for $x > 10$, fulfilling the requirements of Definition 1.

In studying rates of growth, it is useful to note the following proposition which is proved in the exercises.

**Proposition 1.** Suppose that $0 < a < b$. Then

$$x^b > x^a$$

for $x > 1$.

One of the goals in the study of rates of growth is to get information how fast a function grows by comparing it to another function whose growth we understand.
DEFINITION 2. Let $f$ and $g$ be functions. We say that $f$ grows like a multiple of $g$ if there are constants $C > 0$, $D > 0$ and $N$ such that for all $x > N$,

$$Cg(x) < f(x) < Dg(x).$$

(See Figure 2)

Example 2. Show that $f(x) = x^3 - 3x^2 + 5x - 1$ grows like a multiple of $x^3$.

Solution: According to Definition 2, we need to find positive constants $C$, $D$, and $N$ such that

$$Cx^3 < x^3 - 3x^2 + 5x - 1 < Dx^3$$

for all $x > N$.

Deleting negative terms makes a sum larger. Thus, for $x > 1$,

$$x^3 - 3x^2 + 5x - 1 < x^3 + 5x < x^3 + 5x^3 = 6x^3$$

Hence, for the right hand inequality, we may take $D = 6$ and $N = 1$.

Finding $C$ is harder. Deleting positive terms makes a sum smaller. Hence, for $x > 0$,

$$x^3 - 3x^2 + 5x - 1 > x^3 - 3x^2 - 1$$

Subtracting more also makes a quantity smaller. Hence, for $x > 1$,

$$x^3 - 3x^2 - 1 > x^3 - 3x^3 - x^3 = -3x^3$$

Unfortunately, we cannot use $-3$ as $C$ since a positive value is required. To avoid this problem, we replace $3x^2$ and 1 by sufficiently multiples of $x^3$.

We choose $N$ so that both of the following hold for $x > N$:

$$3x^2 < \frac{1}{3}x^3$$

$$1 < \frac{1}{3}x^3$$

The first inequality is valid for $x > 9$ and the second for $x > 3^{1/3} \approx 1.44$. For $x > 9$, both are valid and

$$x^3 - 3x^2 - 1 > x^3 - \frac{1}{3}x^3 - \frac{1}{3}x^3 = \frac{1}{3}x^3$$

Hence we may choose $N = 9$ and $C = 1/3$. This value of $N$ also works for the $D$ inequality with $D = 6$ because $9 > 1$. 
It is a general property of ratios of positive numbers that the larger the denominator, the smaller the number. Thus, for example
\[
\frac{3}{9} < \frac{3}{5}
\]
because 5 < 9. This principle is the basis for the next two examples.

**Example 3.** Find a value of \( m \) such that the following function grows like a multiple of \( x^m \). Prove your answer.

\[
f(x) = \frac{3x^5}{x^3 - 3x^2 + 5x - 1}
\]

**Solution:** We need to find \( C > 0, D > 0, m, \) and \( N \) such that
\[
Cx^m < \frac{3x^5}{x^3 - 3x^2 + 5x - 1} < Dx^m.
\]
for all \( x > N \).

Finding \( m \) is easy. The numerator is a multiple of \( x^5 \) while, from Example 2, the denominator grows like a multiple of \( x^3 \). Hence the whole fraction should grow like a multiple of \( x^2 \), implying \( m = 2 \). More precisely, from the work done in Example 2, we know that for \( x > 9 \)
\[
(5) \quad \frac{1}{3} x^3 < x^3 - 3x^2 + 5x - 1 < 6x^3
\]
We may invert this inequality since, for \( x > 9, x^3 > 0 \). We find
\[
\frac{3}{x^3} > \frac{1}{x^3 - 3x^2 + 5x - 1} > \frac{1}{6x^3}
\]
We may multiply by \( 3x^5 \) since, again, for \( x > 9, x^5 > 0 \). We find
\[
9x^2 > \frac{3x^5}{x^3 - 3x^2 + 5x - 1} > \frac{x^2}{2}
\]
Hence, we may choose \( m = 2, C = 1/2, D = 9 \) and \( N = 9 \).

**Remark:** The principle that increasing the size of the denominator makes the fraction smaller is only valid if both the numerator and denominator are positive. For example
\[
\frac{5}{-7} < \frac{5}{3}
\]
despite the fact that \(-7 < 3\). Thus, the positivity of the \( C \) and \( D \) found in Example 2 was crucial in Example 3.
EXAMPLE 4. Find a value of \( m \) such that the following function grows like a multiple of \( x^m \). Prove your answer.

\[
f(x) = \frac{3x^2 + 1}{x^3 - 3x^2 + 5x - 1}
\]

Solution: We need to find \( C > 0,\ D > 0,\ m, \) and \( N \) such that

\[
Cx^m < \frac{3x^2 + 1}{x^3 - 3x^2 + 5x - 1} < Dx^m
\]

for all \( x > N \).

Solution: Since the denominator grows like a multiple of \( x^3 \) and the numerator like a multiple of \( x^2 \), the whole fraction should grow like a multiple of \( x^2/x^3 \) suggesting that \( m = -1 \). To prove this we find the \( C \) and \( D \)'s for the numerator and denominator. Specifically, from inequality (5), for \( x > 9, \)

\[
\frac{1}{3} x^3 < x^3 - 3x^2 + 5x - 1 < 6x^3
\]

which, upon inversion, becomes.

(6) \[
\frac{3}{x^3} > \frac{1}{x^3 - 3x^2 + 5x - 1} > \frac{1}{6x^3}
\]

For the numerator we find that for \( x > 1 \)

(7) \[
4x^2 > 3x^2 + 1 > 3x^2.
\]

Multiplying inequalities (6) and (7) shows that, for \( x > 9, \)

\[
\frac{12x^2}{x^3} > \frac{3x^2 + 1}{x^3 - 3x^2 + 5x - 1} > \frac{3x^2}{6x^3}
\]

(The multiplication is allowed since, for \( x > 9, \) both inequalities involve only positive numbers.) Hence \( C = \frac{1}{2},\ D = 12,\ N = 9 \) works.

Logarithmic growth is particularly slow. Recall that

\[
\ln x = \int_1^x \frac{1}{t} \, dt
\]

Thus, \( \ln x \) is the area under the curve \( y = 1/x \) between 1 and \( x \). Comparison of areas as in Figure 3 shows that for \( x > 1, \)

(8) \[
\ln x < x
\]

The same inequality holds for \( 1 \geq x > 0 \) since in this range, \( \ln x \leq 0. \) Thus \( \ln x \) grows slower than \( x. \)
Actually, $\ln x$ grows slower than $ax$ for any $a > 0$. For $a > 1$, this follows from (8), while for $0 < a < 1$ this follows from the following proposition which is proved using areas. (See Exercise 15.)

**Proposition 2.** Let $0 < a < 1$. Then

\[(9) \quad \ln x < ax \]

for $x > 4/a^2$.

It is a consequence of Proposition 2 that \textit{logarithmic growth is slower than power growth, regardless of the power, as long as the power is positive}, as the following example demonstrates.

**Example 5.** Find a value of $N$ such that $\ln x < \frac{x^{1/2}}{3}$ for all $x > N$.

**Solution:** We replace $x$ by $x^{1/2}$ in formula (9) finding that for $x^{1/2} > 4/a^2$,

\[
\ln x^{1/2} < ax^{1/2} \\
\frac{1}{2} \ln x < ax^{1/2} \\
\ln x < 2ax^{1/2}
\]

We now let $a = \frac{1}{5}$, finding that $\ln x < \frac{x^{1/2}}{3}$ for $x^{1/2} > 4/a^2 = 144$. Hence, we may use $N = (144)^2$. 

\[\text{Figure 3. } \ln x < x\]
Remark: The value $N = (144)^2 = 20736$ found above is by no means the “best possible” answer. A graphing calculator indicates that in fact $\ln x < \frac{x^{3/2}}{3}$ is true for all $x > 289$. This shows that the value of $N$ given in Proposition 2 can be vastly larger than necessary. This, however, does not matter in determining which function grows faster.

Exponential growth is faster than power growth, as the next example shows.

Example 6. Prove that $2^x$ grows faster than $x^3$.

Scratch work: We must show that there is an $N > 0$ such that

$$x^3 < 2^x$$

for all $x > N$. Since $\ln x$ is an increasing function, our inequality is equivalent with

$$\ln x^3 < \ln 2^x$$

$$3 \ln x < x \ln 2$$

$$\ln x < \frac{\ln 2}{3} x$$

which, according to Proposition 2, with $a = \frac{\ln 2}{3}$, is true for

$$x > \frac{36}{(\ln 2)^2} \approx 74.93.$$ 

We cannot, however, use 74.93 as the value of $N$ since this value is only an approximation. If the actual value of $36/(\ln 2)^2$ is slightly greater than 74.93, then it is conceivable that inequality (10) might fail for some $x > 74.93$. However, presuming that our calculator has at least 2 decimal accuracy, we can be certain that, say, $80 > \frac{36}{(\ln 2)^2}$. Hence, we may use $N = 80$.

Solution: Assume that $x > 80$. Then

$$36/(\ln 2)^2 < x$$

Thus, from Proposition 2,

$$\ln x < \frac{\ln 2}{3} x$$

$$3 \ln x < x \ln 2$$

$$\ln x^3 < \ln 2^x$$
Since $e^x$ is an increasing function we may continue this sequence of inequalities as follows:

\[ e^{\ln x^3} < e^{\ln 2^x} \]
\[ x^3 < 2^x \]

as desired.

Remarks: Note that the formal solution required the fact that the exponential function is increasing; not that the logarithm function is increasing, as our “scratch work” had suggested. Typically, if we apply a particular function to both sides of an inequality in the “scratch work,” then in the formal solution we will apply the corresponding inverse function to both sides of an inequality. Fortunately, it is a general principal that the inverse of an increasing function is increasing. (See Exercise 12 below.) Similar comments apply for decreasing functions. (See Exercise 13 below.)

Example 6 also demonstrates that in rounding computed values of $N$, we should never “round down.” If a particular inequality is known to true for, say, all $x > 4.01$, then it might not be valid for all $x > 4$. It will, however, hold for all $x > 5$.

Another general principle is that if $f(x)$ grows more slowly than $g(x)$ which grows more slowly than $h(x)$, then $f(x)$ grows more slowly than $h(x)$. (See Exercise 10.) We apply this principle in the next example.

Example 7. Find a value of $N$ such that $7x^2 < 2^x$ for all $x > N$.

Scratch work: If we attempt to solve this using the same idea as in Example 6, we take the log of the inequality obtaining

\[ \ln(7x^2) < \ln 2^x \]
\[ \ln 7 + \ln x^2 < x \ln 2 \]
\[ \ln 7 + 2 \ln x < x \ln 2 \]

Unfortunately, this cannot be transformed into something to which Proposition 2 applies.

Solution: We reason that $7x^2$ grows more slowly than $x^3$ which grows more slowly than $2^x$. Specifically, $7x^2 < x^3$ for $x > 7$ and, from
the solution to Example 6, \(x^3 < 2^x\) for \(x > 80\). Hence, \(7x^2 < 2^x\) for \(x > 80\).

**Example 8.** Find an \(a > 0\) such that the following function grows like a multiple of \(a^x\). Prove your answer.

\[
\frac{2^x + x \ln x + x^3}{4^x + x^3 3^x + 1}
\]

**Solution** The fastest growing term in the numerator and denominator are, respectively, \(2^x\) and \(4^x\). Hence, we expect that the fraction should grow like \(2^x/4^x = (1/2)^x\), suggesting that we may use \(a = 1/2\). To prove our answer, we must find constants \(C > 0\), \(D > 0\) and \(N\) such that

\[
(11) \quad C 2^{-x} < \frac{2^x + x \ln x + x^3}{4^x + x^3 3^x + 1} < D 2^{-x}
\]

for all \(x > N\).

We first consider the denominator. Since \(x^3 3^x\) and 1 should grow more slowly than \(4^x\), there should exist an \(N\) such that for \(x > N\),

\[
1 < 4^x
\]

\[
x^3 3^x < 4^x
\]

The first inequality is true for all \(x > 0\). Dividing by \(3^x\) and using the fact that \(\ln x\) is an increasing function, we see that the second inequality is equivalent with

\[
x^3 < \left(\frac{4}{3}\right)^x
\]

\[
3 \ln x < x \ln \frac{4}{3}
\]

\[
\ln x < \left(\frac{1}{3} \ln \frac{4}{3}\right) x
\]

which, from Proposition 2, holds for \(x > 4/((1/3 \ln 4/3)^2) \approx 434.9876\). Hence, for, say, \(x > 450\)

\[
4^x < 4^x + x^3 3^x + 1 < 4^x + 4^x + 4^x = 3(4^x)
\]

Inverting, we see

\[
(12) \quad \frac{1}{3(4^x)} < \frac{1}{4^x + x^3 3^x + 1} < \frac{1}{4^x}
\]
For the numerator, we note that for $x > 1$,
\[
2^x + x \ln x + x^3 < 2^x + x \cdot x + x^3
\]
\[
= 2^x + x^2 + x^3
\]
\[
< 2^x + 2x^3
\]
where we used $\ln x < x$ in the first line.

On the other hand, from Example 6, $x^3 < 2^x$ for $x > 80$. Hence, for such $x$,
\[
2^x + x \ln x + x^3 < 3(2^x).
\]

Inequalities (12) and (13) will both hold for $x > 450$. Multiplying these inequalities shows that inequality (11) holds for all $x > 450$ with $C = 1/3$ and $D = 3$, finishing the proof.

The rate of growth of the sum of two functions is determined by the fastest growing term in the sum. The situation is more complicated in the case of products. For example, $x^3 3^x$ grows faster than either $3^x$ or $x^3$. However, since exponential growth is faster than power growth, $x^3 3^x$ grows slower than $a^x$ for any $a > 3$.

**Example 9.** Prove that $x^3 3^x$ grows more slowly than $(3.1)^x$.

**Solution** We must show that there is an $N$ such that for all $x > N$
\[
x^3 3^x < (3.1)^x
\]

However, since $\ln x$ is an increasing function, this equation is equivalent with:
\[
x^3 < \left(\frac{3.1}{3}\right)^x
\]
\[
3 \ln x < x (\ln 3.1 - \ln 3)
\]
which, from Proposition 2, holds for
\[
x > \frac{36}{(\ln 3 - \ln 3.1)^2} \approx 33482.99.
\]

Thus, we can be certain that the stated inequality holds for, say, $x > 40,000$.

The next example is based on this idea.
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Example 10. Find positive numbers $C$, $D$, $a$, $b$, and $N$ such that

$$Ca^x < \frac{2^x + x \ln x + x^3}{x^3 3^x + 1} < Db^x$$

for all $x > N$.

Solution From the solution to Example 9, equation (14) holds for $x > 40,000$. Hence, for such $x$

$$3^x < x^3 3^x + 1 < (3.1)^x + (3.1)^x = 2(3.1)^x.$$ 

Inverting:

$$\frac{1}{2(3.1)^x} < \frac{1}{x^3 3^x + 1} < \frac{1}{3^x}.$$ 

Multiplication by inequality (13) (which holds for $x > 450$) yields the estimate

$$\frac{2^x}{2(3.1)^x} < \frac{2^x + x \ln x + x^3}{x^3 3^x + 1} < 3 \frac{2^x}{(3^x)} \frac{1}{2 \left(\frac{2}{3.1}\right)^x} < \frac{2^x + x \ln x + x^3}{x^3 3^x + 1} < 3 \left(\frac{2}{3}\right)^x$$

which is true for $x > 40,000$. Thus, we may choose $a = 2/3.1$, $b = 2/3$, $C = 1/2$, $D = 3$, and $N = 40,000$. In place of 3.1 we could, of course use any number strictly greater than 3, although different choices result in different values of $N$. We cannot use 3.

Exercises

(1) Suppose that $0 < a < b$.

(a) Prove that if $x > 1$, then $x^a > 1$. Hint: Use calculus to prove that, $y = x^a$ is increasing on $(0, \infty)$.

(b) Prove that if $x > 1$, then $x^b > x^a$. Hint: From (a), $x^{b-a} > 1$.

(2) For each function $f(x)$, find a value of $m$ such that $f(x)$ grows like a multiple of $g(x) = x^m$. Prove your answer by finding constants $C$, $D$ and $N$ fulfilling the requirements of
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Definition 2 on page 33.

(a) \[ f(x) = x^4 + 3x^2 + 1 \]
(b) \[ f(x) = 5x^2 - 3x + 7 \]
(c) \[ f(x) = \frac{5x^2 - 3x + 7}{x^4 + 3x^2 + 1} \]
(d) \[ f(x) = \frac{x^4 + 3x^2 + 1}{5x^2 - 3x + 7} \]
(e) \[ f(x) = x^4 + 3x^2 - 5x + 1 \]
(f) \[ f(x) = 5x^2 - 3x - \sqrt{x} + 7 \]
(g) \[ f(x) = \frac{5x^2 - 3x - \sqrt{x} + 7}{x^4 + 3x^2 - 5x + 1} \]
(h) \[ f(x) = \frac{x^4 + 3x^2 - 5x + 1}{5x^2 - 3x - \sqrt{x} + 7} \]
(i) \[ f(x) = \frac{x^5 - 4x^2 - 5}{x^5 + 14x - 3} \]
(j) \[ f(x) = \frac{x^5 + 14x - 3}{x^5 - 4x^2 - 5} \]

(3) In each part, find constants \( C > 0, D > 0 \) and \( N \) such the stated inequalities are valid for all \( x > N \). State in words what each inequality tells you about rates of growth.

(a) \[ C \left( \frac{2}{3} \right)^x < \frac{2x + 14x - 3}{3x - 4x^2 - 5} < D \left( \frac{2}{3} \right)^x \]
(b) \[ C \left( \frac{1.5}{3} \right)^x < \frac{2x^2 + 14x - 3}{3x - 4x^2 - 5} < D \left( \frac{2.5}{3} \right)^x \]
(c) \[ C \left( \frac{5}{4} \right)^x < \frac{5x - 3x - \sqrt{x} + 7}{4x + 3x^2 - 5x + 1} < D \left( \frac{5}{4} \right)^x \]
(d) \[ C \left( \frac{4.9}{4} \right)^x < \frac{5x^3 - 3x - \sqrt{x} + 7}{4x^2 + 3x^2 - 5x + 1} < D \left( \frac{5.1}{4} \right)^x \]

(4) Find a value of \( N \) such that for \( x > N \),
(a) \( x^2 - 3x + 27 < x^2 \)
(b) \( x^2 - 3x + 27 > .9x^2 \)
(c) Explain why there is no \( N \) such that \( x^2 - 3x + 27 > x^2 \) for all \( x > N \).

(5) Find a value of \( N \) such that for \( x > N \),
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(a) \( x^3 + 3x^2 - 27x + 1 < (1.01)x^3 \)
(b) \( x^3 + 3x^2 - 27x + 1 > x^3 \)
(c) Explain why there is no \( N \) such that \( x^3 + 3x^2 - 27x + 1 < x^3 \) for all \( x > N \).

(6) For each pair of functions below (i) determine which is the faster growing, (ii) prove your answer by finding a number \( N \) fulfilling the requirements of Definition 1 on page 31, and (iii) use a graphing calculator (or other graphing technology) to estimate the smallest usable value for \( N \).

Warning: In (ii) you must PROVE that the stated value of \( N \) works. The graph produced in (iii) is NOT sufficient proof since, for example, the calculator display shows only a portion of the complete graph of the function. In fact, all computer generated graphs really plot only a finite number of points. What the function actually does in between the plotted points cannot be determined from the display.

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<td>( 1, \quad \frac{3x^3 \ln x}{x^3 - 3x^2 + 5x - 1} )</td>
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(7) (a) Prove that for \( n > 3, \ n! > \frac{2}{5}(3^n) \). Hint \( 3! = 3 \cdot 2 = (3^3)\frac{2}{5}. \) \( 5! = 5 \cdot 4 \cdot 3! > 3 \cdot 3 \cdot 3! = (3^3)\frac{2}{5}. \) Repeat the same argument in the general case.
(b) Find an \( N \) such that \( \frac{2}{5}3^n > 2^n \) for \( n > N \). How does it follow that \( n! \) grows faster than \( 2^n \)?
(c) Use the reasoning from (a) to find an \( N \) such that \( n! > \frac{3}{32}4^n \) for \( n > N \). Use this to find an \( N \) such that \( n! > 3^n \) for all \( n > N \).

(8) For each pair of functions below (i) determine which is the faster growing, (ii) prove your answer by finding a number \( N \) fulfilling the requirements of Definition 1 on page 31, and
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(iii) use a graphing calculator (or other graphing technology) to estimate the smallest usable value for $N$.

**Warning:** See the comments in the instructions for Exercise 6.

(a) $x^2, \ e^x$
(b) $x^2 \ln x, \ (1.5)^x$
(c) $3^x, \ x^3 2^x$
(d) $\frac{x^3}{2^x}, \ \frac{1}{(1.5)^x}$
(e) $\frac{1}{(1.5)^x}, \ \frac{x^2 \ln x}{2^x}$
(f) $\frac{3^n}{n!}, \ \frac{1}{2^n}$
(g) $\frac{3^n + n^2 + 1}{2^n} - 5, \ \frac{3^n}{2^n}$
(h) $\frac{n!}{100^n}, \ (1.1)^n$

(9) Are there positive constants $C$ and $N$ such that

$$Cx < \ln x$$

for all $x > N$. If so, find them. If not, prove that such constants don’t exist.

(10) Suppose that $f(x)$, $g(x)$, and $h(x)$ are functions such that $f(x)$ grows faster than $g(x)$ and $g(x)$ grows faster than $h(x)$. Use Definition 1 on page 31 to prove that then $f(x)$ grows faster than $h(x)$.

(11) Let $f$ be an invertible function and let $g = f^{-1}$ be the inverse function. (Hence, $g(f(x)) = x$ for all $x$ in the domain of $f$.) Prove that $f(g(y)) = y$ for all $y$ in the range of $f$. **Hint:** Since $y$ is in the range of $f$, $y = f(x)$ for some $x$.

(12) Let $f$ be an increasing, invertible function and let $g = f^{-1}$. Prove that $g$ is also increasing. **Hint** Suppose that there are numbers $a < b$ in the domain of $g$ such that $g(a) \geq g(b)$. What do you know about the effect of applying $f$ to inequalities?
(13) State (carefully) a version of Exercise 12 that applies to decreasing functions. Then solve your exercise.

(14) In the notes we used integrals to prove that for \( x > 0 \), \( \ln x < x \).

(a) Prove this by using calculus to find the minimum for the function \( f(x) = x - \ln x \). Use the second derivative test to prove that the value you find really is a minimum.

(b) Prove that \( \ln x \leq (1/e)x \) for all \( x > 0 \). Why is this inequality false with \( \leq \) replaced by \( < \)? *Hint:* Use the same idea as you used in (a).

(c) Use the result from (b) to prove that \( \ln x < \sqrt{x} \) for all \( x > 0 \). *Hint:* Apply (b) with \( x \) replaced by \( x^{1/2} \).

(15) In this series of exercises, you prove Proposition 2.

(a) Let \( b > 1 \). Use a **clearly labeled** diagram similar to Figure 1 to prove that for \( x > b \),

\[
\ln x - \ln b < \frac{x}{b}
\]

*Hint:* Integrate from \( b \) to \( x \).

(b) Use the result from (a) to prove that for \( x > b \)

\[
\ln x < \frac{x}{b} + b
\]

(c) Use the result from (b) to prove that if \( x > b^2 \), then

\[
\ln x < \frac{2x}{b}
\]

*Hint:* \( x > b^2 \) is equivalent with \( \frac{x}{b} > b \). Proposition 2 follows by letting \( b = 2/a \).