MA 301 Practice Questions for Test 3, Fall 2004

You should bring a calculator to the test to be able to do
problems such as Problem 10

(1) State the “official” definition of “\( \lim_{x \to a} f(x) = L \).”

(2) Prove, using well labeled diagrams, the following version of
Theorem 2 in the text. Be careful to bring in the role of the
Bounded Increasing Theorem and the decreasing nature of \( f \)
into your proof. (The proof is given at the end of this review
sheet.)

**Theorem (2’).** Suppose \( a_n > 0 \) for all \( n \) and \( f(x) \) is an
integrable, decreasing function on \([0, \infty)\) such that \( a_n = f(n) \)
for all \( n \in \mathbb{N} \). Then \( s = \sum_{n=1}^{\infty} a_n \) exists if

\[
\int_0^{\infty} f(x) \, dx < \infty
\]

(3) Prove, using well labeled diagrams, the following version of
Theorem 4 in the text. Be careful to bring in the role of the
decreasing nature of \( f \) into your proof. (The proof is given
at the end of this review sheet.)

**Theorem (4’).** Suppose \( a_n > 0 \) for all \( n \) and \( f(x) \) is an
integrable, decreasing function on \([0, \infty)\) such that \( a_n = f(n) \)
for all \( n \in \mathbb{N} \). Then

\[
s_n \geq \int_1^{n+1} f(x) \, dx
\]

(4) **Use the integral test** to prove that the following series
converges. Then write a sum that expresses \( s \) to within
\( \pm 10^{-5} \). **Hint:** To evaluate the integral, make the substitution \( u = x^4 + 5 \).

\[
s = \sum_{n=1}^{\infty} \frac{4n^3}{(n^4 + 5)^2}
\]

(5) Prove, using \( M \), that the following series diverges.

\[
\sum_{n=1}^{\infty} \frac{1}{n^{1/3}}
\]
(6) Classify each of the following series as either (a) conditionally convergent or (b) absolutely convergent or (c) divergent. Prove your answers.

(a) \[ \sum_{1}^{\infty} \frac{(-1)^n(\sqrt{3n} + 2)}{n^3 + 5n + 5} \]

(b) \[ \sum_{1}^{\infty} \frac{(-1)^n7^n}{(7.1)^n + n} \]

(c) \[ \sum_{1}^{\infty} \frac{(-1)^nn^27^n}{8^n + n} \]

(d) \[ \sum_{1}^{\infty} \frac{(-1)^nn^2}{\sqrt{n^3} + 3} \]

(e) \[ \sum_{1}^{\infty} \frac{(-1)^n(3n + 1)}{\sqrt{n^5} + 3} \]

(7) Prove that the following series converges.

\[ \sum_{1}^{\infty} \frac{n(\ln n)^2}{n^3 + 3} \]

(8) Let

\[ s = \sum_{1}^{\infty} \frac{n^2 + \sqrt{\ln n + 1}}{n^4 + 3n + 7} \]

(a) Prove that this series converges.
(b) Write a sum which computes \( s \) to within \( \pm 10^{-3} \).

(9) Prove that \( Z = 1/(3\pi + 5) \) is irrational. You may assume that \( \pi \) is irrational. You MAY NOT use Proposition 1 from Chapter 9.

(10) Find

(a) an \textit{explicit} irrational number \( Z \) satisfying \( 17/13 < Z < 18/13 \). You need not prove that \( Z \) is irrational.
(b) an \textit{explicit} rational number \( Z \) satisfying \( \pi < Z < 22/7 \).

(11) Find an explicit one-to-one correspondence between the sets \( A \) and \( B \) where:

(a) \( A = (-1, 3) \) and \( B = (0, 1) \).
(b) \( A \) is the set of even natural numbers and \( B \) is the set of odd natural numbers.
(c) $A$ is the set of natural numbers which are multiples of 2 and $B$ is the set of natural numbers which are multiples of 3.

Various Results from the Text

**Theorem (2').** Suppose $a_n > 0$ for all $n$ and $f(x)$ is an integrable, decreasing function on $[0, \infty)$ such that $a_n = f(n)$ for all $n \in \mathbb{N}$. Then $s = \sum_{1}^{\infty} a_n$ exists if $\int_{0}^{\infty} f(x) \, dx < \infty$.

**Proof** Each $a_n$ is the length of a line segment drawn from the point $(n, 0)$ on the $x$-axis to the graph of $y = f(x)$ as in Figure 1.

![Figure 1. Theorems 2' and 4'](image)

The area of a rectangle of width one having this line segment as its right edge is $a_n$. (See Figure 2). This rectangle also lies entirely below the graph of $y = f(x)$ since this graph is decreasing. Since the left side of the first rectangle extends to $x = 0$,

\begin{equation}
(1) \quad s_n = a_1 + a_2 + \cdots + a_n \leq \int_{0}^{n} f(x) \, dx \leq \int_{0}^{\infty} f(x) \, dx.
\end{equation}

Finally, since the $a_n$ are all positive, $s_n$ is an increasing sequence. From the Bounded Increasing Theorem, $\lim s_n$ either exists or equals $\infty$. Formula (1) proves that the limit is not $\infty$. Hence the limit exists, proving the convergence of the sum. \qed

**Theorem (4').** Suppose $a_n > 0$ for all $n$ and $f(x)$ is an integrable, decreasing function on $[0, \infty)$ such that $a_n = f(n)$ for all $n \in \mathbb{N}$. Then

$$s_n \geq \int_{1}^{n+1} f(x) \, dx$$
Proof Each $a_n$ is the length of a line segment drawn from the point $(n, 0)$ on the $x$-axis to the graph of $y = f(x)$ as in Figure 1. The area of a rectangle of width one having this line segment as its left edge is $a_n$. (See Figure 3). This rectangle also lies entirely above the graph of $y = f(x)$ since this graph is decreasing.

Since the right side of the $n$th rectangle extends to $x = n + 1$,

(2) \[ s_n = a_1 + a_2 + \cdots + a_n \geq \int_1^{n+1} f(x) \, dx. \]

proving Theorem 4’.