Solution

MATH 351
November 8, 2013
Quiz 7

1. Let \( \mathcal{B} = \{ [1, 2]^t, [1, -3]^t \} \).

\( \text{(a) (3 points)} \) Find the point matrix \( P_\mathcal{B} \).

\( \text{(b) (4 points)} \) Find the coordinate matrix \( C_\mathcal{B} \).

\( \text{(c) (4 points)} \) Use (b) to find the \( \mathcal{B} \)-coordinates of \((-4, 7)^t\).

Solution:

(a) The point matrix is simply the matrix whose columns are the ordered basis \( \mathcal{B} \), so
\[
P_\mathcal{B} = \begin{bmatrix} 1 & 1 \\ 2 & -3 \end{bmatrix}.
\]

(b) We know \( C_\mathcal{B} = P_\mathcal{B}^{-1} \), so
\[
C_\mathcal{B} = \begin{bmatrix} 1 & 1 \\ 2 & -3 \end{bmatrix}^{-1} = \frac{1}{1 \cdot (-3) - 2 \cdot 1} \begin{bmatrix} -3 & -1 \\ -2 & 1 \end{bmatrix} = \frac{1}{-5} \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}.
\]

(c) We know if \( X' \) are the \( \mathcal{B} \)-coordinates of \( X \) then \( X' = C_\mathcal{B} X \), so
\[
\frac{1}{5} \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} -4 \\ 7 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \end{bmatrix}.
\]
2. (9 points) Recall $P_2 = \{a_0 + a_1x + a_2x^2\}$ is the vector space of all polynomials of degree less than or equal to 2. Let $\mathcal{B} = \{p_1(x), p_2(x), p_3(x)\}$ be any basis for $P_2$. Suppose $L : P_2 \to \mathbb{R}^3$ is given by $L(ap_1(x) + bp_2(x) + cp_3(x)) = [a, b, c]^T$. You may assume $L$ is a linear transformation. Show $L$ is an isomorphism of $P_2$ onto $\mathbb{R}^3$.

There are several ways to do this, and here are three of them.

**Method 1:** To say $L$ is an isomorphism is the same as saying $L$ is invertible. If we note that $S : \mathbb{R}^3 \to P_2$ given by $S([a, b, c]^T) = ap_1(x) + bp_2(x) + cp_3(x)$ is a well defined function, and

$$S \circ L(ap_1(x) + bp_2(x) + cp_3(x)) = S([a, b, c]^T) = ap_1(x) + bp_2(x) + cp_3(x),$$

which is the identity map on $P_2$, so $S = L^{-1}$, and therefore, $L$ is an isomorphism.

**Method 2:** Since $\dim \mathbb{R}^3 = 3 = \dim P_2$, there is a theorem in the book which says $L$ is an isomorphism if and only if the nullspace of $L$ is $\{0\}$. Suppose $p(x) \in P_2$ with $p(x) = ap_1(x) + bp_2(x) + cp_3(x)$. Note, since $\mathcal{B}$ is a basis of $P_2$, then by definition, $L(ap_1(x) + bp_2(x) + cp_3(x)) = [0, 0, 0]^T$, if and only if $a - b - c = 0$. But then $p(x) = 0p_1(x) + 0p_2(x) + 0p_3(x) = 0$ (the zero polynomial). Thus, $p(x)$ is in the nullspace of $L$ if and only if $p(x) = 0$. Thus, $L$ is an isomorphism.

**Method 3:** Let $M$ be the matrix of $L$ with respect to the ordered bases $\mathcal{B}$ of $P_2$ and the standard basis $\mathcal{B} = \{(1, 0, 0)^T, (0, 1, 0)^T, (0, 0, 1)^T\}$. Then, since $L(p_1(x)) = [1, 0, 0]^T$, $L(p_2(x)) = [0, 1, 0]^T$, and $L(p_3(x)) = [0, 0, 1]^T$, we have

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I,$$

the $3 \times 3$ identity matrix. Since $M$ is invertible, $L$ is invertible.
Theorem 3: (Uniqueness Theorem) The determinant is the only function $D : M(n, n) \rightarrow \mathbb{R}$ with the properties

(a) $D(I) = 1$
(b) $D$ satisfies the row exchange, scalar, and the additive row properties.

Consequence:

Theorem 4: For any $n \times n$ matrices $A$ and $B$,

$$\det(AB) = \det A \cdot \det B$$

Proof: First suppose $\det B = 0$. Then $B$ is not invertible, so $\text{rank} B < n$. So now $\text{rank}(AB) \leq \text{rank} B < n$, by the Rank of Products Theorem.

So $AB$ is not invertible, so $\det(AB) = 0$, so $\det(AB) = \det A \cdot \det B$.

So now assume $\det B \neq 0$.

Let $D : M(n, n) \rightarrow \mathbb{R}$ be given by

$$D(A) = \frac{\det(AB)}{\det B}.$$

Use the uniqueness theorem to show $D(A) = \det A$, i.e., show $D(I) = 1$ and the three row properties hold.

Note $D(I) = \frac{\det(I \cdot B)}{\det B} = \frac{\det B}{\det B} = 1$.

We need to show the three properties hold:
Recall, if \( A_1, A_2, \ldots, A_n \) are the rows of \( A \), then
\[
AB = \begin{bmatrix}
A_1 \\
A_2 \\
\vdots \\
A_n
\end{bmatrix} B = \begin{bmatrix}
A_1 B \\
A_2 B \\
\vdots \\
A_n B
\end{bmatrix}
\] (i)

(a) Row exchange property. Suppose \( M \) is obtained by interchanging 2 rows of \( A \). \( \det M = -\det A \).

By (i) \( M \cdot B \) is obtained from \( AB \) by switching the same two rows. So \( \det (M B) = -\det (AB) \)

\[
D(M) = \frac{\det (M B)}{\det B} = -\frac{\det (AB)}{\det B} = -D(A).
\]

(b) Suppose \( M \) is obtained from \( A \) by multiplying the \( j \)th row of \( A \) by a scalar \( c \). Then by (i) \( M \cdot B \) is obtained from \( AB \) by multiplying the \( j \)th row by \( c \).

So \( \det (M B) = c \cdot \det (AB) \)

Hence, \( D(M) = \frac{\det (M B)}{\det B} = c \cdot \frac{\det (AB)}{\det B} = c \cdot D(A) \).

Recall if \( A_1 = U + V \), then
\[
\det \begin{bmatrix}
U \\
A_2 \\
\vdots \\
A_n
\end{bmatrix} = \det \begin{bmatrix}
U \\
A_2 \\
\vdots \\
A_n
\end{bmatrix} + \det \begin{bmatrix}
V \\
A_2 \\
\vdots \\
A_n
\end{bmatrix}
\]

So now \( AB = \begin{bmatrix}
(U + V) B \\
A_2 B \\
\vdots \\
A_n B
\end{bmatrix} \)

So \( \det AB = \det \begin{bmatrix}
U B \\
A_2 B \\
\vdots \\
A_n B
\end{bmatrix} + \det \begin{bmatrix}
V B \\
A_2 B \\
\vdots \\
A_n B
\end{bmatrix} \)
So \( D(A) = \frac{\det(AB)}{\det B} = \)

\[
\frac{1}{\det B} \left[ \det \left[ \begin{array}{c} VB \\ A_2 B \\ \vdots \\ A_n B \end{array} \right] + \det \left[ \begin{array}{c} VB \\ A_2 B \\ \vdots \\ A_n B \end{array} \right] \right] = \]

\[
\frac{\det \left[ \begin{array}{c} VB \\ A_2 B \\ \vdots \\ A_n B \end{array} \right]}{\det B} + \frac{\det \left[ \begin{array}{c} VB \\ A_2 B \\ \vdots \\ A_n B \end{array} \right]}{\det B} = \]

\[
D\left( \left[ \begin{array}{c} 0 \\ A_1 \\ \vdots \\ A_n \end{array} \right] \right) + D\left( \left[ \begin{array}{c} A_2 \\ 0 \\ \vdots \\ A_n \end{array} \right] \right). \]

So \( D \) satisfies (c).

The additive row property. So \( D(A) = \det A \)

Thus \( \frac{\det(AB)}{\det B} = \det A \), so \( \det(AB) = \det A \cdot \det B \).

**Thm.5:** For any \( n \times n \) matrix \( A \)

\[
\det(A) = \det(A^T),
\]

Let \( A = \left[ \begin{array}{c} A_1 \\ A_2 \end{array} \right] \) be a \( 2 \times 2 \) matrix.

Consider the parallelogram with sides \( A_1 \) and \( A_2 \).

\[
A = \left[ \begin{array}{c} 3 \\ 1 \\ 0 \\ 2 \end{array} \right]
\]

\( \det A = 6 = \text{area } [A_1, A_2] \)

**Prop.** Let \( \text{area } [A_1, A_2] \) be the area of the (possibly degenerate) parallelogram with sides \( A_1, A_2 \). Then \( \det A = \text{area } [A_1, A_2] \).
Proof. We claim that \( \text{Area}(A_1, A_2) \) satisfies

1. \( \text{Area}(I) = 1 \)
2. \( \text{Area}[A_1, A_2] = \text{Area}[A_2, A_1] \)
3. \( \text{Area}[cA_1, A_2] = c \text{Area}[A_1, A_2] \)
4. \( \text{Area}[A_1 + cA_2, A_2] = \text{Area}[A_1, A_2] \).

The proof of claim (2) is trivial, as is

(4) \( c \).

Use the fact that

\[ c h_2 = h_1 \]

by similarity of triangles.