Recall: We established several properties of determinants:

1. Row exchange property
2. Scalar property
3. Additive row property

Theorem 2: An $n \times n$ matrix $A$ is invertible if and only if $\det A \neq 0$.

Lemma: If $A$ and $B$ are row equivalent $n \times n$ matrices, then $\det A = \alpha \det B$, for some scalar $\alpha \neq 0$. [Follows from (1), (2), (3)].

Proof of Theorem 2: Let $R$ be the reduced row echelon form of $A$. Then by the lemma, $\alpha \det R = \det A$ for some $\alpha \neq 0$. If $A$ is invertible, then $R$ is the $n \times n$ identity matrix. We've seen $\det R = 1$, so $\det A = \alpha \neq 0$.

Now suppose $A$ is not invertible. Then $R$ has a row of zeros (since rank $A < n$). (In fact we know $n^{th}$ row must be $[0 \ 0 \ \ldots \ \ 0]$.) So expand $\det R$ in this last row, to see $\det R = 0$. So by the lemma, $\det A = \alpha \det R = \alpha \cdot 0 = 0$. So if $\det A \neq 0$, $A$ must be invertible. Q.E.D.

Now consider $\det : M(n, \mathbb{R}) \to \mathbb{R}$ as a function (transformation).
Thm 3: (Uniqueness Theorem) The determinant is the only function \( D : M(n \times n) \rightarrow \mathbb{R} \) with the properties

(a) \( D(I) = 1 \)

(b) \( D \) satisfies the row exchange, scalar, and the additive row properties.

Consequence:

Thm 4: For any \( n \times n \) matrices, \( A \) and \( B \),

\[
\det(AB) = \det A \cdot \det B
\]

Proof: First suppose \( \det B = 0 \). Then \( B \) is not invertible, so \( \text{rank } B \leq n \). So now \( \text{rank}(AB) \leq \text{rank } B < n \), by The Rank of Products Theorem. So \( AB \) is not invertible, so \( \det(AB) = 0 \), so \( \det(AB) = \det A \cdot \det B \).

So now assume \( \det B \neq 0 \)

Let \( D : M(n \times n) \rightarrow \mathbb{R} \) be given by

\[
D(A) = \det(A \cdot B) / \det B.
\]

Use the uniqueness theorem to show \( D(A) = \det A \).

i.e. Show \( D(I) = 1 \) and the three row properties hold.

Note \( D(I) = \frac{\det (I \cdot B)}{\det B} = \frac{\det B}{\det B} = 1 \)

We'll finish the proof on Monday.