and \( \cos kt \) represent possible displacement functions for the mass. In fact, such linear combinations are the only twice-differentiable solutions to equation (1.6); hence, all displacement functions are of this form. Similarly, equation (1.6) describes the motion of the mass in the presence of friction. These issues are discussed in greater length in Section 2.2.

38. \( \checkmark \) Let

\[
A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}
\]

(a) Use the definition of matrix addition to prove that the only \( 2 \times 2 \) matrix \( B \) such that \( A + B = A \) is the zero matrix. The point of this problem is that one should think of \( A + 0 = A \) as the defining property of the zero matrix.

(b) Use the definition of matrix addition to prove that the only \( 2 \times 2 \) matrix \( B \) such that \( A + B = 0 \) is

\[
B = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}
\]

The point of this problem is that one should think of \( A + (-A) = 0 \) as the defining property of \(-A\).

39. Prove vector space properties (b), (e), (g), (h), and (i) for \( M(m, n) \).

40. Let \( B, C, \) and \( X \) be elements of some vector space. In the following discussion, we solved the equation \( 3X + B = C \) for \( X \). At each step we used one of the vector space properties. Which property was used? \( \text{[Note: We define } \]

\[
C - B = C + (-B).
\]

\[
3X + B = C
\]

\[
(3X + B) + (-B) = C + (-B)
\]

Properties (a) and (e)

\[
3X + (B + (-B)) = C - B
\]

Definition of \( C - B \) and property (?)

\[
3X + 0 = C - B
\]

Property (?)

\[
3X = C - B
\]

Property (?)

\[
\frac{1}{3}(3X) = \frac{1}{3}(C - B)
\]

Property (?)

\[
\frac{1}{3}X = \frac{1}{3}(C - B)
\]

Property (?)

\[
X = \frac{1}{3}(C - B)
\]

Property (?)

41. Let \( X \) and \( Y \) be elements of some vector space. Prove, putting in every step, that \( -(2X + 3Y) = (-2)X + (-3)Y \). You may find Proposition 2 useful.

42. \( \checkmark \) Let \( X, Y, \) and \( Z \) be elements of some vector space. Suppose that there are scalars \( a, b, \) and \( c \) such that \( aX + bY + cZ = 0 \). Show that if \( a \neq 0 \), then

\[
X = \left( -\frac{b}{a} \right) Y + \left( -\frac{c}{a} \right) Z
\]

Do your proof in a step-by-step manner, to demonstrate the use of each vector space property needed. \( \text{[Note: In a vector space, } X + Y + Z \text{ denotes } X + (Y + Z).] \)

43. Prove that in any vector space, if \( X + Y = 0 \), then \( Y = -X \). (Begin by adding \(-X\) to both sides of the given equality.)

44. Prove Proposition 2. \( \text{[Hint: From Exercise 43, it suffices to prove that } X + (-1)X = 0.\]}

1.1.1 Computer Projects

Our goal in this discussion is to plot some elements of the span of the vectors \( A = [1, 1] \) and \( B = [2, 3] \) using MATLAB. Before we begin, however, let us make a few general comments. When you start up MATLAB, you will see something like \( >> \) followed by a blank line. If the instructions ask you to enter \( 2 + 2 \), then you should type \( 2 + 2 \) on the screen behind the \( >> \) prompt and then press the enter key. Try it!

\[
>> 2+2
\]

\[
an =
\]
Entering matrices into MATLAB is not much more complicated. Matrices begin with “[” and end with “]”. Entries in rows are separated with either commas or spaces. Thus, after starting MATLAB, our matrices $A$ and $B$ would be entered as shown. Note that MATLAB repeats our matrix, indicating that it has understood us.

```plaintext
>> A = [1 1]
A =
   1  1
>> B = [1 3]
B =
   1  3
```

Next we construct a few elements of the span of $A$ and $B$. If we enter “2*A+B”, MATLAB responds

```plaintext
ans =
   3  5
```

(Note that * is the symbol for “times.” MATLAB will complain if you simply write $2A+B$.)

If we enter $(-5)*A + 7*B$, MATLAB responds

```plaintext
ans =
   2  16
```

Thus, the vectors [3, 5] and [2, 16] both belong to the span.

We can get MATLAB to automatically generate elements of the span. Try entering the word “rand”. This should cause MATLAB to produce a random number between 0 and 1. Enter “rand” again. You should get a different random number. It follows that entering the command $C$=rand*A+rand*B should produce random linear combinations of $A$ and $B$. Try it!

To see more random linear combinations of these vectors, push the up-arrow key. This should return you to the previous line. Now you can simply hit “enter” to produce a new random linear combination. By repeating this process, you can produce as many random elements of the span as you wish.

Next, we will plot our linear combinations. Begin by entering the following lines. Here “figure” creates a figure window, “hold on” tells MATLAB to plot all points on the same graph, and “axis([-5.5,-5.5])” tells MATLAB to show the range $-5 \leq x \leq 5$ and $-5 \leq y \leq 5$ on the axes. The command “hold on” will remain in effect until we enter “hold off”:

```plaintext
>> figure
>> hold on
>> axis([-5.5,-5.5])
```

A window (the Figure window) showing a blank graph should pop up.

Points are plotted in MATLAB using the command “plot.” For example, entering `plot(3,4)` will plot the point $(3, 4)$. Return to the MATLAB Command window and try plotting a few points of your own choosing. (To see your points, you will either need to return to the Figure window or move and resize the Command and Figure windows so that you can see them both at the same time. Moving between windows is accomplished by pulling down the Window menu.) When you are finished, clear the figure window by entering “cla” and then enter the following line:

```plaintext
C=rand*A+rand*B; plot(C(1),C(2))
```

This will plot one point in the span. [ $C(1)$ is the first entry of $C$ and $C(2)$ is the second.] You can plot as many points as you wish by using the up-arrow key as before.

**Exercises**

1. Plot the points $[1, 1], [1, -1], [-1, 1],$ and $[-1, -1]$ all on the same figure. When finished clear the figure window by entering the “cla” command.

2. Enter the vectors $A$ and $B$ from the discussion above.
   
   (a) Get MATLAB to compute several different linear combinations of them. (Reader’s choice.)
   
   (b) Use $C$=rand*A+rand*B to create several “random” linear combinations of $A$ and $B$.
   
   (c) Plot enough points in the span of $A$ and $B$ to obtain a discernible geometric figure. Be patient. This may require plotting over 100 points. What kind of geometric figure do they seem to form? What are the coordinates of the vertices?

   **Note:** If your patience runs thin, you might try entering the following three lines. The “;” keeps Matlab from echoing the command every time it is being executed.

   ```plaintext
   for i=1:200
   C=rand*A+rand*B; plot(C(1),C(2));
   end
   ```

   This causes MATLAB to execute any commands between the “for” and “end” statements 200 times.

   (d) The plot in part (c) is only part of the span. To see more of the span, enter the commands:

   ```plaintext
   for i=1:200
   C=2*rand*A+rand*B; plot(C(1),C(2),'r');
   end
   ```
The “r” in the plot command tells MATLAB to plot in red.

3. Describe in words the set of points $s \cdot A + t \cdot B$ for $-2 \leq s \leq 2$ and $-2 \leq t \leq 2$. Create a MATLAB plot that shows this set reasonably well. Use yet another color. (Enter “help plot” to see the choice of colors.) [Hint: “rand” produces random numbers between 0 and 1. What would “rand-0.5” produce?]

4. In Exercise 13 on page 16, it was stated that each element of the span of $X$ and $Y$ satisfies $5x + 3y - 2z = 0$.

(a) Check this by generating a random matrix $C$ in the span of $X$ and $Y$ and computing $5\cdot C(1) + 3\cdot C(2) - 2\cdot C(3)$. Repeat with another random element of the span.

(b) Plot a few hundred elements of this span in $\mathbb{R}^3$. Before doing so, close the Figure window by selecting Close from the File menu. Next, enter “figure”, then “axis([-4,-4,-4,-4,-4])”, and “hold on”. A three-dimensional graph should pop up. The command plot3(C(1),C(2),C(3)) plots the three-dimensional vector $C$.

Describe the geometric figure so obtained. What are the coordinates of the vertices? Why is this to be expected?

1.1.2 Applications to Graph Theory I

Figure 1.10 represents the route map of an airline that serves four cities, A, B, C, and D. Each arrow represents a daily flight between the two cities.

The information from this diagram can be represented in tabular form, where the numbers represent the number of daily flights between the cities:

<table>
<thead>
<tr>
<th>from/to</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>B</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>C</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>D</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>

The $4 \times 4$ matrix obtained by deleting the labels from the preceding table is what we refer to as the route matrix. The route matrix could be stored in a computer. One could then, for example, use this information as the basis for a computer program to find the shortest connection for a customer.

Route maps are examples of what are called **directed graphs**. In general, a directed graph is a finite set of points (called **vertices**), together with arrows connecting some of the vertices. A directed graph may be described using a matrix just as was done for route maps. Specifically, if the vertices are $V_i$, $V_2$, ..., $V_n$, then the graph will be represented by the matrix $A$, where $a_{ij}$ is the number of arrows from $V_i$ to $V_j$.

Graph theory may also be applied to anthropology. Suppose an anthropologist is studying generational dominance in an extended family. The family members are M (mother), F (father), S1 (first born son), S2 (second born son), D1 (first born daughter), D2 (second born daughter), MGM (maternal grandmother), MGF (maternal grandfather), PGM (paternal grandmother), and PGF (paternal grandfather). The anthropologist represents the dominance relationships by a directed graph where an arrow is drawn from each individual to any individual he or she directly dominates.

In the exercises you will study the dominance relationship given in Figure 1.11.

We will say more about the matrix of a graph in Section 3.2.

**Self-Study Questions**

1. Give the route matrix for each of the route maps in Figure 1.12.

2. For each of the following matrices, draw a route map that could correspond to the given matrix:

   $$(a) \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad (b) \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$