Homework due Friday, 9/2/7
Pg. 82 # 37, 40, 43
Pg. 107 # 1(b), c), 3(a), f'), G, 9, 12, 13, 17, 21, 23, 28, 30, 31, 32, 29
Pg. 91 - computer projects 1, 2, 3
Pg. 121 1, 2

Exam: "Approximate Grades"

85 - 100 A
74 - 84 B
61 - 73 C
52 - 60 D

Median: 66
5. (12 points) Find a $3 \times 4$ matrix $A$ with no entries equal to zero for which the equation $AX = B$ is solvable if and only if $B$ lies in the span of $[1, 2, 1]^t$ and $[-1, 1, 3]^t$.

By a theorem in the book, $AX = B$ is solvable if and only if $B$ lies in the column space of $A$. So we need 4 columns whose span is that of $[1, 2, 1]^t$ and $[-1, 1, 3]^t$.

$$A = \begin{bmatrix} 1 & -1 & 1 & 2 \\ 2 & 1 & 5 & 6 \\ 1 & 3 & 5 & 3 \end{bmatrix}.$$  

$$x = \begin{bmatrix} x_1, x_2, x_3, x_4 \end{bmatrix}^t.$$  Then

$$AX = x_1 A_1 + x_2 A_2 + x_3 A_3 + x_4 A_4.$$  Where $A_i$ are the four columns of $A$. 


6. **(12 points)** Show a $2 \times 2$ matrix with linearly dependent rows must have linearly dependent columns.

**Pf:** Let \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) with linearly dependent rows. We have either 
\[ \begin{bmatrix} c \\ d \end{bmatrix} = s \begin{bmatrix} a \\ b \end{bmatrix} \text{ or } \begin{bmatrix} a \\ b \end{bmatrix} = t \begin{bmatrix} c \\ d \end{bmatrix}. \]
By switching rows, we assume \( \begin{bmatrix} c \\ d \end{bmatrix} = s \begin{bmatrix} a \\ b \end{bmatrix}. \)

**Case 1:** \( a = 0 \). So \( A = \begin{bmatrix} 0 & b \\ 0 & sb \end{bmatrix}, \)
so the columns are linearly dependent.

**Case 2:** \( a \neq 0 \). So \( A = \begin{bmatrix} a & b \\ sa & sb \end{bmatrix}. \)

Note \( b = \frac{b}{a} \cdot a \). Then \( sb = s \cdot \left( a \cdot \frac{b}{a} \right) = \frac{a}{b} \left( sa \right). \) So \( \begin{bmatrix} b \\ sb \end{bmatrix} = \frac{b}{a} \begin{bmatrix} a \\ sa \end{bmatrix} \), so
the columns are linearly dependent.
Recall from last time:

Prop 1: Let \( V \) be a vector space, and suppose \( A_k \) is a linear combination of \( A_1, A_2, \ldots, A_k \). Then \( \text{Span}\{A_1, A_2, \ldots, A_k\} = \text{Span}\{A_1, \ldots, A_{k-1}\} \).

Def: A basis for a vector space \( V \) is a linearly independent subset \( B = \{A_1, \ldots, A_n\} \) which spans \( V \).

Ex: \( A_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ A_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \ A_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \); \( A_4 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \) (\( A_4 = 2A_2 + A_3 \))

\( \text{Span}\{A_1, A_2, A_3, A_4\} = \text{Span}\{A_1, A_2, A_3\} \).

\[
\begin{bmatrix}
1 & -1 & 1 \\
0 & 2 & 1 \\
1 & 0 & 1
\end{bmatrix} \xrightarrow{\text{RREF}}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\( A_1, A_2, A_3 \) are linearly independent. Then \( B = \{A_1, A_2, A_3\} \) is a basis for \( \text{Span}\{A_1, A_2, A_3, A_4\} \).

Recall. If \( A_1, A_2, \ldots, A_n \) is a collection of matrices, to determine whether they're linearly independent, form a dependency system from:

\[
x_1A_1 + x_2A_2 + \cdots + x_nA_n = 0
\]

If \( x_i \) is a pivot variable we say \( A_i \) is a pivot matrix for the collection. Then (i) all non-pivot matrices are linear combinations of the pivot matrices, and all pivot matrices are linearly independent.

Thm: (3) If \( A_1, A_2, \ldots, A_n \) is a collection of \( mxn \) matrices, then the
Theorem 4: For an \( n \times n \) matrix, the pivot columns form a basis for the column space.

\[
\begin{pmatrix}
  2 & -1 & 1 \\
  1 & 3 & 5 \\
\end{pmatrix} \rightarrow \begin{pmatrix}
  10 & 2 \\
  0 & 0 & 0 \\
\end{pmatrix}
\]

So \( \begin{pmatrix} 2 \\ 1 \end{pmatrix} \) and \( \begin{pmatrix} -1 \\ 3 \end{pmatrix} \) form a basis for the column space of \( A \).

Theorem 5: If \( A \) and \( B \) are row equivalent \( n \times n \) matrices with columns \( A_1, A_2, \ldots, A_n \) and \( B_1, B_2, \ldots, B_n \), respectively.

Suppose, for some \( j \), we have

\[
B_j = c_1 B_1 + c_2 B_2 + \cdots + c_{j-1} B_{j-1} + c_{j+1} B_{j+1} + \cdots + c_n B_n
\]

Then

\[
A_j = c_1 A_1 + c_2 A_2 + \cdots + c_{j-1} A_{j-1} + c_{j+1} A_{j+1} + \cdots + c_n A_n
\]

Proof:

We can rewrite (1) as

\[
c_1 B_1 + c_2 B_2 + \cdots + c_{j-1} B_{j-1} - B_j + c_{j+1} B_{j+1} + \cdots + c_n B_n = 0
\]

or

\[
B X = 0, \text{ where } X = [c_1, c_2, \ldots, c_{j-1}, -1, c_{j+1}, \ldots, c_n]'
\]

So \( X \) is in the null space of \( A \). But then

\[
A X = 0 \text{ as well, which is equivalent to (2)}
\]
Testing sets of functions for linear independence

(3) \( \cos(2x) = \cos^2x - \sin^2x = 1 - 2\sin^2x \)

(3) says \( \cos(2x) + 2\sin^2x - 1 = 0 \)
So \( \{\cos(2x), \sin^2x, 1\} \) are linearly dependent.

How about \( \{e^x, e^{2x}, 1\} \)?

Is there a choice of \(a, b, c\) (not all 0) for which \(Ae^x + be^{2x} + c \cdot 1 = 0\) (4)?

Note evaluate at \(x=0\), \(a + b + c = 0\) (5)

Differentiate (4)

\(ae^x + 2be^{2x} + 0 = 0\)

Evaluate at \(x=0\), \(a + 2b = 0\) (6)

Repeat the process. \(ae^x + 4be^{2x} = 0\)

\(a + 4b = 0\) (7)

\[
\begin{align*}
    a + b + c &= 0 \\
    a + 2b &= 0 \\
    a + 4b &= 0
\end{align*}
\]

\(\Rightarrow a = b = c = 0\)