Recall: Thm: For any m×n matrix \( A \), the non-zero rows of any echelon form \( R \) of \( A \) form a basis for the row space of \( A \).

Ex: We can use this to determine bases for subspaces. Find a basis for the subspace \( W \) of \( \mathbb{R}^5 \) spanned by

\[
\begin{align*}
Y_1 &= \begin{bmatrix} 3 \\ 2 \\ -1 \\ -5 \\ 1 \end{bmatrix}, \\
Y_2 &= \begin{bmatrix} 2 \\ -1 \\ 4 \\ -3 \\ 1 \end{bmatrix}, \\
Y_3 &= \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \\
Y_4 &= \begin{bmatrix} 5 \\ -3 \\ 11 \\ -3 \\ 0 \end{bmatrix}, \\
Y_5 &= \begin{bmatrix} -2 \\ -3 \\ -8 \\ 5 \\ 4 \end{bmatrix}
\end{align*}
\]

\[
A = \begin{bmatrix} 3 & 2 & -1 & -1 & -5 \\ 2 & -1 & 4 & -3 & 3 \\ 1 & -1 & 0 & 1 & 0 \\ -3 & 5 & -1 & 11 & 0 \\ -2 & 3 & -8 & 5 & 4 \end{bmatrix}
\]

Gaussian Elimination:

\[
\begin{bmatrix} 1 & 1 & -1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\]

So

\[
Y_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \\
Y_2 = \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix}, \\
Y_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]

is a basis for \( W \).

Theorem 2: (Rank Theorem) For any m×n matrix \( A \), the column space and row space have the same dimension, namely \( \text{rank}(A) \).

Proof: Let \( R \) be an echelon form of \( A \). By non-zero rows theorem, the non-zero rows of \( R \) form a basis for the row space of \( A \). By definition \( \text{rank}(A) \) is the number of non-zero rows of \( R \). So the row space has dimension equal to \( \text{rank}(A) \).

On the other hand, the pivot columns of \( A \) form a basis for the column space. Since there is one pivot in each non-zero row of \( R \), there are \( \text{rank}(A) \) pivot columns.

So the column space of \( A \) has dimension \( \text{rank}(A) \). QED

Theorem 3: For any m×n matrix \( A \),

\[
\text{rank}(A^t) = \text{rank}(A^t).
\]

Proof: The rows of \( A^t \) are the columns of \( A \).

So the row space of \( A^t \) is the column space of \( A \).
So \( \text{rank } A = \text{rank } A^T \) by Rank Theorem.

Example:
\[
\begin{bmatrix}
1 & 2 & 3 & 4 & 7 \\
-1 & 0 & 4 & 2 & 1 \\
3 & 1 & & & -1 & \Pi
\end{bmatrix} = A
\]

\[
A^T = \begin{bmatrix}
1 & -1 & 3 & 7 \\
2 & 0 & 4 & 6 \\
3 & 1 & 2 & -1 \\
4 & 1 & 1 & \Pi
\end{bmatrix}
\]

The columns of \( A \) are linearly independent if and only if \( \text{rank } A = n \). The rows of \( A \) are linearly independent if and only if \( \text{rank } A = n \).

Theorem 4: Let \( A \) be an \( m \times n \) matrix.

The columns of \( A \) are linearly independent if and only if \( \text{rank } A = n \). The rows of \( A \) are linearly independent if and only if \( \text{rank } A = m \).

Theorem 5: Let \( A \) be an \( m \times n \) matrix.

The equation \( A X = B \) is solvable for every \( B \in \mathbb{R}^m \) if and only if \( \text{rank } A = m \).

Proof:
Recall \( A X = B \) is solvable \( \iff \) \( B \) lies in the column space of \( A \). So \( A X = B \) is solvable for all \( B \in \mathbb{R}^m \) \( \iff \) the column space of \( A \) is all of \( \mathbb{R}^m \). So if \( A X = B \) is solvable for all \( B \), then \( \text{rank } A = m \).

Conversely, if \( \text{rank } A = m \) then by the Rank Theorem, the column space of \( A \) is \( m \)-dimensional. So \( A \) has \( m \) linearly independent columns, which then must be a basis for \( \mathbb{R}^m \). So for every \( B \in \mathbb{R}^m \), \( B \) lies in the column space of \( A \), so \( A X = B \) is solvable.

Definition: For an \( m \times n \) matrix \( A \), the dimension of the null space is called the nullity of \( A \), denoted \( \text{null } A \) or \( \text{null } (A) \).
Theorem 6: (Rank and Nullity) For any $m \times n$ matrix $A$, $\text{rank } A + \text{null } A = n$.

Proof: The nullspace of $A$ is the solution set to $A\mathbf{x} = \mathbf{0}$. The number of non-pivot columns is the number of free variables in this solution set. Therefore, $\text{null } A$ is this number of non-pivot columns. On the other hand, the pivot columns form a basis for the column space, and there are $\text{rank } A$ such columns (Rank Theorem). Every column is either a pivot or non-pivot column. So the number of all columns is the sum of the number of these two types, i.e. $n = \text{rank } A + \text{null } A$. $\square$

Theorem 7: Let $A$ be an $m \times n$ matrix. The equation $A\mathbf{x} = \mathbf{B}$ has at most one solution if and only if $\text{rank } A = n$.

Proof: We proved there is at most one solution if and only if the nullspace of $A$ is $\{\mathbf{0}\}$, so if and only if $\text{null } A = 0 \iff \text{rank } A = n$, by Rank and Nullity Theorem. $\square$