RECALL: (Monday) We proved the Translation Theorem: If \( A \in M(m,n), B \in M(n,1), T \in M(n,n) \) and \( A \cdot T = B \), then the solution set to
\[
A \mathbf{x} = B
\]
is all vectors of the form \( \mathbf{x} = T + \mathbf{y} \), with \( A \mathbf{y} = \mathbf{0} \).

The solution set to the homogeneous equation \( A \mathbf{x} = \mathbf{0} \) is (def) the nullspace of \( A \) and is a subspace of \( \mathbb{R}^n \).

**Ex:** Let \( A = \begin{bmatrix} -1 & 3 & -5 & 4 \\ 2 & -3 & 7 & -4 \\ -2 & 1 & 3 & 5 \end{bmatrix} \). Find the nullspace of \( A \).

**Solution:** The Reduced Row Echelon Form of \( A \) is
\[
\begin{bmatrix} 1 & 0 & -2 & 2 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\]
so let \( [x_1, x_2, x_3, x_4]^t \) be in the nullspace of \( A \). Then \( x_3, x_4 \) are the free variables, and
\[
x_2 = x_3 - 2x_4, \quad x_1 = -2x_3 - 2x_4,
\]
so
\[
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}
\]
Note \( \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \) are linearly independent.

Subspaces of \( \mathbb{R}^3 \):
(i) \( \mathbb{R}^3 \), (ii) line through the origin, (iii) plane through the origin, (iv) \( \mathbb{R}^3 \).
Thm 5: Let \( A \) be an \( m \times n \) matrix, the spanning vectors for the system \( AX = 0 \) span the nullspace of \( A \).

Thm 6: (Subspace Properties)
A subset \( W \) of a vector space \( V \) is a subspace if and only if:
1. For any \( X, Y \in W \), \( X + Y \in W \)
2. For any \( X \in W \), and \( a \in \mathbb{R} \), \( aX \in W \)
3. \( 0 \in W \).

Pf: Homework

Ex: \( V = \mathcal{C}(\mathbb{R}) = \{ f: \mathbb{R} \to \mathbb{R} \} \).
\( W = \{ f \text{ such that } f \text{ is continuous on } (0,1) \} \).
If \( f, g \) are continuous on \( (0,1) \), is \( f + g \) continuous? For any \( c \), is \( cf \) continuous? Yes (Calc. Thm).
\( W \) is a subspace of \( \mathcal{C}(\mathbb{R}) \).
\( U = \{ f \mid f(\pi) = 0 \} \). Is \( U \) a subspace of \( \mathcal{C}(\mathbb{R}) \)?
Note, \( 0 \) is in \( U \). If \( f, g \in U \) and \( c \in \mathbb{R} \), then
1. \((f+g)(\pi) = f(\pi) + g(\pi) = 0 + 0 = 0 \) so \( f + g \in U \)
2. \((cf)(\pi) = c(f(\pi)) = c \cdot 0 = 0 \). Yes!
Is \( W = \{ f \mid f(1) = 17 \} \) a subspace?

Prop. 3 Any subspace \( W \) of a vector space \( V \) is itself a vector space with the inherited operations from \( V \).

[Recall the following from the lecture 8-23]

\[ V = M(m,n) \]

Theorem: \( \text{a) Given } X, Y, Z \in V \)

a) \( X + Y \) is well defined
b) \( X + Y = Y + X \)
c) \( X + (Y + Z) = (X + Y) + Z \)
d) There is an element \( 0 \) so that
   \( X + 0 = X \) for all \( X \)
e) There is an element \( -X \) so that
   \( X + (-X) = 0 \)

2. If \( a, b \in R \)
   f) \( aX \)
   g) \( (ab)X = a(bX) \)
   h) \( a(X + Y) = aX + aY \)
   i) \( (a+b)X = aX + bX \)
   j) \( 0 \cdot X = X \)

Any set \( V \) with an \textit{addition} \( + \)
and \textit{scalar multiplication}, which satisfies
(a) - (i) is called a vector space.
Ch. 2: When is a set linearly independent?

Theorem (Test for Linear Independence)
Let \( S = \{A_1, A_2, \ldots, A_k\} \) be a subset of a vector space \( V \). Then \( S \) is a linearly independent set if and only if
\[
x_1 A_1 + x_2 A_2 + \cdots + x_k A_k = 0
\]
has only the solution \( x_1 = x_2 = \cdots = x_k = 0 \).
If $a, b \in \mathbb{R}$

1. $aX$
2. $(a+b)X = aX + bX$
3. $a(X+Y) = aX + aY$
4. $(a+b)X = aX + bX$
5. $1 \cdot X = X$

Any set $V$ with an addition $+$ and scalar multiplication, which satisfies $(S1)-(S3)$ is called a vector space.