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On the construction of G-spaces and applications to homogeneous spaces

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INTRODUCTION

In (3), the author defined the notion of a G-space. A G-space is a weaker notion than that of an H-space. The main purpose of this paper is to present various means of constructing G-spaces. As an application of some of the techniques of (3) and of this paper (though not an application of the concept of G-space) we shall prove the following theorem:

THEOREM. *Let G be a connected compact Lie group and let H be a connected subgroup of maximal rank. Then $H_3(G/H; Z) = 0$. In fact, the Hurewicz homomorphism is trivial for odd dimensions.*

This paper is divided into five sections. The first section is devoted to restating the definition of a G-space and reviewing some properties of G-spaces. In section 2 we show that the n-connective covering of a G-space is a G-space. In section 3 we show how to form two-stage Postnikov systems which are G-spaces. The next section, section 4, is devoted to the construction of finite dimensional G-spaces. This is done by considering homogeneous spaces, an idea which Jerrold Siegel first used in (6) to find a finite dimensional G-space which is not an H-space. For the case when T is a toroidal subgroup of a Lie group G, we show that G/T is a G-space if and only if $\pi_2(G/T) = 0$. The last section, section 5, is devoted to the proof of the theorem mentioned in the previous paragraph. This section is independent of the rest of the paper, with the exception of some lemmas in section 4.

1. PRELIMINARIES

We shall always assume that we are dealing with spaces homotopic to CW complexes. Consider the set of all maps $F: X \times S^n \to X$ such that $F|X$ is the identity on X. Now $F|S^n$ gives us a subset of base point preserving maps of $S^n \to X$. The set of homotopy classes of these maps is a subgroup $G_n(X, *)$ of the nth homotopy group $\pi_n(X, *)$. We call this subgroup the *nth evaluation subgroup of* $\pi_n(X)$.

There is another definition of $G_n(X)$. Let $X^X =$ the space of maps from X to X with the compact open topology. Let $\omega: X^X \to X$ be the evaluation map $f \to f(*)$. Then $G_n(X, *)$ may be defined as the image of $\pi_n(X^X, 1_X)$ under ω_*. That is,

$$G_n(X) = \omega_*[\pi_n(X^X, 1_X)]. \tag{1}$$

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Now a connected CW-complex X is a *G-space* if $G_n(X) = \pi_n(X)$ for all n. Thus a G-space is characterized by the property that any map $X \vee S^n \to X$ may be extended to a map $X \times S^n \to X$ for all n.

We summarize below the known results about G-spaces:

PROPOSITION 1·1

(*a*) *Any connected H-space is a G-space* (3).

(*b*) *Every Whitehead Product vanishes in a G-space* (6).

(*c*) *Every higher order spherical Whitehead Product contains zero in a G-space* (6).

(*d*) *Let X be a G-space with finitely generated homology. If X is not contractible, then the Euler–Poincaré number* $\chi(X) = 0$. *Also* $\pi_2(X)$ *has no torsion-free element* (3).

(*e*) *Let X be a simply connected G-space with finitely generated homology. Then the rational cohomology ring is the tensor product of exterior algebras with odd dimensional generators.* (Haslam (8).)

We now describe Theorem 6·3 of (3). Let X be a CW-complex. Corresponding to any element $\alpha \in G_n(X)$, there is a homomorphism $\bar{\lambda}: H^i(X;G) \to H^{i-n}(X;G)$ for all i and any coefficient ring G. We write $\bar{\lambda}$ as acting on the right, i.e. $\bar{\lambda}: x \to x\bar{\lambda}$. We have the following formula

$$(u \cup v)\bar{\lambda} = u \cup v\bar{\lambda} + (-1)^{n \cdot \dim v} u\bar{\lambda} \cup v. \tag{2}$$

Now suppose $p: E \to X$ is a principal fibration with fibre of type $(\pi, m-1)$ (π Abelian). This fibration corresponds to an element $u \in H^m(X;\pi)$.

Now we quote Theorem 6·3 of (3) using the above notation:

PROPOSITION 1·2. *Suppose* $F: X \times S^n \to X$ *is such that* $F|S^n = \alpha$ *and* $F|X = 1_X$. *Then there exists a map* $\tilde{F}: E \times S^n \to E$ *such that* $\tilde{F}|E = 1_E$ *and*

$$\begin{array}{ccc} E \times S^n & \xrightarrow{\tilde{F}} & E \\ {\scriptstyle p\times 1}\downarrow & & \downarrow{\scriptstyle p} \\ X \times S^n & \xrightarrow{F} & X \end{array} \quad \textit{commutes}$$

if and only if $u\bar{\lambda} = 0$.

We consider one more fact, which is of use in sections 3 and 4.

PROPOSITION 1·3. *Given a map* $\phi: A \to X$ *and a map* $\Phi: A \times X \to X$ *such that* $\Phi|A = \phi$ *and* $\Phi|X = 1_X$, *then we have* $\phi_*: \pi_i(A) \to G_i(X)$ *for all i.*

Proof. Consider the map

$$F: X \times S^j \xrightarrow{1 \times f} X \times A \xrightarrow{\Phi} X.$$

Now $F|X = 1_X$ and $[F|S^j] = [\Phi \circ f] = [\phi \circ f] = \phi_*[f]$. Hence $\phi_*[f] \in G_j(E)$.

2. COVERING SPACES OF G-SPACES

In this section we shall record that an n-connective covering of a G-space is a G-space. The proof is an application of Proposition 1·2. Then we shall note that any covering space of a G-space is a G-space.

THEOREM 2·1. *An n-connective covering space of a G-space is a G-space.*

Proof. Suppose X is $n-1$ connected. Let $\tilde{X} \xrightarrow{p} X$ be the n-connective covering for X. Then $\tilde{X} \xrightarrow{p} X$ corresponds to an element $u \in H^n(X; \pi_n(X))$. Now every $\alpha \in G_{n+i}(X)$ gives rise to a homomorphism $\bar{\lambda}$: $H^*(X; \pi_n(X)) \to H^{*-n-i}(X; \pi_n(X))$ and by dimensional considerations $u\bar{\lambda} = 0$. Thus for every $\alpha \in G_{n+i}(X)$, there is an $\tilde{\alpha} \in G_{n+i}(\tilde{X})$ such that $p_*(\tilde{\alpha}) = \alpha$. Since p_* is an isomorphism for $i > 0$, we have that $G_{n+i}(\tilde{X}) = \pi_{n+i}(\tilde{X})$ for all $i > 0$.

THEOREM 2·2. *Any covering space of a G-space is a G-space.*

Proof. The proof follows from the observation that for any covering space $\tilde{X} \xrightarrow{p} X$, we have $p_*(\alpha) \in G_n(X)$ only if $\alpha \in G_n(\tilde{X})$. (See Theorems 6·1 and 6·2 of (3).) Since X is a G-space, we are finished.

3. Two-stage Postnikov systems

If we wish to use Proposition 1·2 to construct a G-space by beginning with a $K(\pi, n)$ and building up a Postnikov system, we shall see that the problem becomes very complicated. This is because we have to check that the k-invariant vanishes under every possible $\bar{\lambda}$. We shall, however, demonstrate a method for creating G-spaces out of two-stage Postnikov systems.

Let p be a prime number and consider the Eilenberg–MacLane space $K(Z_p, 2n)$. Let $\alpha \in H^{2n}(K(Z_p, 2n); Z_p) = Z_p$ be a generator. Then $\alpha^r \neq 0$ for all r. Let p: $E \to K(Z_p, 2n)$ be a principal fibration with k-invariant α^{kp} (for integer $k > 0$). So

$$\alpha^{kp} \in H^{2nkp}(K(Z_p, 2n); Z_p)$$

and the fibre of p is a $K(Z_p, 2nkp-1)$.

THEOREM 3·1. *E is a G-space.*

Proof. E has two non-zero homotopy groups, $\pi_{2n}(E) = \pi_{2nkp-1}(E) = Z_p$. We shall use Proposition 1·2 to show that $G_{2n}(E) = \pi_{2n}(E)$. First note that $K(Z_p, 2n)$ is an H-space. Let $X = K(Z_p, 2n)$. Then we can find a map F: $X \times S^{2n} \to X$ so that $f = F|S^{2n}$ represents any element of $G_{2n}(X) = \pi_{2n}(X) = Z_p$. In particular, we choose F so that $f^*(\alpha)$ is a generator of $H^{2n}(S^{2n}; Z_p)$. Then F gives rise to a $\bar{\lambda}$: $H^*(X; Z_p) \to H^{*-2n}(X; Z_p)$ such that $\alpha\bar{\lambda} = 1 \in H^0(X; Z_p)$, see (3), section 5. Then, by induction since $\bar{\lambda}$ is a derivation, $(\alpha^{kp})\bar{\lambda} = kp\alpha^{kp-1} = 0$ since $p \equiv 0 \in Z_p$. Thus by Proposition 1·3 there is a $\tilde{F}$: $E \times S^{2n} \to E$ such that $\tilde{f} = \tilde{F}|S^{2n}$ represents a class $[\tilde{f}] \in G_{2n}(E)$. Since

$$p_*[\tilde{f}] = [p\tilde{f}] = [f] \neq 0,$$

then $[\tilde{f}] \neq 0$ and so generates $\pi_{2n}(E) = Z_p$. Thus $G_{2n}(E) = \pi_{2n}(E)$.

The other half of the proof consists of showing that $G_{2nkp-1}(E) = \pi_{2nkp-1}(E)$. Since i: $K(Z_p, 2nkp-1) \to E$ induces an isomorphism on the $2nkp-1$ dimensional homotopy groups, the result follows from the following lemma.

LEMMA 3·2. *Let p: $E \to B$ be a principal fibration with fibre F. If i: $F \to E$ is the inclusion map, then $i_*(\pi_j(F)) \subseteq G_j(E)$ for all j.*

Proof. Because $p: E \to B$ is a principal fibration there is a map $\Phi: F \times E \to E$ such that $\Phi | E = 1_E$ and $\Phi | F = i$. The result follows immediately by Proposition 1·3.

Under most circumstances, E is not an H-space. In fact, E is an H-space if and only if α^{kp} is primitive. That is, if μ is the multiplication of $K(Z_p, 2n)$, we see that E is an H-space if and only if $\mu^*(\alpha^{kp}) = \alpha^{kp} \otimes 1 + 1 \otimes \alpha^{kp}$. Now

$$\begin{aligned} \mu^*(\alpha^{kp}) &= (\mu^*\alpha)^{pk} = (1 \otimes \alpha + \alpha \otimes 1)^{pk} \\ &= (1 \otimes \alpha^p + \alpha^p \otimes 1)^k \\ &= (1 \otimes \alpha^{kp} + \alpha^{kp} \otimes 1) + k\alpha^p \otimes \alpha^{p(k-1)} + \dots. \end{aligned}$$

Now $k\alpha^p \otimes \alpha^{p(k-1)} = 0$ only if k is a multiple of p. By an induction type argument we see that E is an H-space if and only if k is a power of p.

These results were independently discovered by Harold Haslam (8).

4. CONSTRUCTION OF FINITE DIMENSIONAL G-SPACES

We shall first discuss a method of constructing finite dimensional G-spaces due to Jerrold Siegel (6), who used it to find a compact G-space which is not an H-space. This result answers a question of Porter's (5), for it provides a compact space all of whose higher order spherical Whitehead products contain zero, and yet the space is not an H-space.

In this section and in the next, G will denote a compact Lie group and H a closed subgroup of G. We shall consider the fibration $H \xrightarrow{i} G \xrightarrow{\rho} G/H$. The underlying observation for this section and the next is the following lemma.

LEMMA 4·1. $\rho_*: \pi_i(G) \to G_i(G/H) \subseteq \pi_i(G/H)$.

Proof. Apply Proposition 1·3 taking $A = G$, $X = G/H$, $\phi = p$ and $\Phi: G \times G/H \to G/H$ given by $\Phi(g, xH) = gxH$.

COROLLARY 4·2. *$G_i(G/H) = \pi_i(G/H)$ if $\rho_*: \pi_i(G) \to \pi_i(X)$ is onto, or equivalently, if $i_*: \pi_{i-1}(H) \to \pi_{i-1}(G)$ is injective.*

Now let H be a closed toroidal subgroup, T. Since T is aspherical, $\rho_*: \pi_i(G) \to \pi_i(G/T)$ is onto for $i > 2$, and also since T is connected, ρ_* is onto for $i = 1$. This follows from the fibre homotopy exact sequence of the fibration $\rho: G \to G/T$. Thus, by virtue of the above corollary, $G_i(G/T) = \pi_i(G/T)$ for $i \neq 2$. Thus G/T is a G-space if and only if $\pi_2(G/T) = G_2(G/T)$.

THEOREM 4·3. *With notation of the preceding paragraph, G/T is a G-space if and only if $\pi_2(G/T) = 0$.*

Proof. First assume $\pi_2(G/T) = 0$. In this case $G_2(G/T) = \pi_2(G/T)$ so by the previous paragraph, G/T is a G-space.

Conversely assume that $\pi_2(G/T) \neq 0$. We know $\pi_2(G) = 0$ because G is a Lie group, and we know that $\pi_1(T)$ is a free Abelian group. Then from the fibre homotopy exact sequence we have $0 \to \pi_2(G/T) \to \pi_1(T)$, so $\pi_2(G/T)$ is a free Abelian group. Now according to Proposition 1·1 (d), G/T is not a finite dimensional G-space because $\pi_2(G/T)$ has torsion-free elements. This proves the theorem.

5. Applications to homogeneous spaces

The applications we have in mind are based on Corollary 4·2 and the following proposition which is a summary of some results of (3).

First we need some notation. Let $h\colon \pi_i(X) \to H_i(X; Z)$ be the Hurewicz homomorphism. We denote by h_p the Hurewicz homomorphism mod p; that is, the composition

$$\pi_i(X) \xrightarrow{h} H_i(X; Z) \longrightarrow H_i(X; Z_p).$$

We consider p to be a prime or $p = \infty$, by which we mean Z_∞ = rational numbers.

Proposition 5·1. *Suppose X has finitely generated homology.*

(*a*) *Then $G_{2i}(X) \subseteq \ker h_\infty$.*

(*b*) *If, in addition, $\chi(X) \neq 0$, then $G_{2i+1}(X) \subseteq \ker h_p$ for all primes p and ∞.* (See (3), Theorems 4·1 and 5·1.)

We shall consider the following situation. Let G be a compact, connected Lie group. Let H be a closed connected subgroup of G. We assume that H has the same rank as G. (The rank of a Lie group is the dimension of the maximal connected Abelian subgroup T contained in G, and T is always a torus called the maximal torus of G.) The homogeneous spaces, G/H, have been studied by many mathematicians, starting with H. Hopf, H. Samelson, H. C. Wang, A. Borel, to mention a few. A great deal has been discovered about the homology groups of G/H by using the Serre Spectral Sequence, classifying spaces, Lie algebras and other methods. One key theorem in the subject is the following (1), (2).

Theorem (Bott–Borel). *Let T be a maximal toral subgroup of a connected, compact Lie group G. Then $H^*(G/T; Z)$ has no torsion and $H^r(G/T; Z) = 0$ for odd r.*

Now, as a corollary to this theorem we obtain the following lemma.

Lemma 5·2. *Let T be a maximal torus of a compact connected Lie group G. Then the Hurewicz homomorphism $h\colon \pi_n(G/T) \to H_n(G/T; Z)$ is trivial for all $n \neq 2$.*

Proof. From the fibre homotopy exact sequence for the fibration $\rho\colon G \to G/T$, using the fact that T is connected and aspherical, we see that $\rho_*\colon \pi_n(G) \to \pi_n(G/T)$ is onto for $n \neq 2$. Thus, by Corollary 4·2, $G_n(G/T) = \pi_n(G/T)$.

If n is odd, then $H_n(G/T) = 0$ by the Bott–Borel Theorem. Hence the Hurewicz map is trivial for odd dimensions.

If n is even and $n \neq 2$, then $H_n(G/T)$ is a free Abelian group (or the trivial group). By Proposition 5·1*a*, $G_n(G/T) \subset \ker h_\infty = \ker h$. Since $G_n(G/T) = \pi_n(G/T)$, we see that h is the zero homomorphism.

Theorem 5·3. *Let H be a connected subgroup of maximal rank in a connected, compact Lie group G. Then the Hurewicz homomorphism of G/H is zero for odd dimensions.*

Proof. The subgroups $T \subset H \subset G$ give rise to a fibration $p\colon G/T \to G/H$ with fibre H/T. This, in turn gives rise to the exact ladder

$$(*)\qquad \begin{array}{ccccccccc} \cdots \longrightarrow & \pi_n(G/T) & \longrightarrow & \pi_n(G/T, H/T) & \xrightarrow{d} & \pi_{n-1}(H/T) & \longrightarrow \cdots \\ & \downarrow{\scriptstyle h_2} & & \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle h_1} & \\ \cdots \longrightarrow & H_n(G/T) & \longrightarrow & H_n(G/T, H/T) & \xrightarrow{\partial} & H_{n-1}(H/T) & \longrightarrow \cdots \end{array}$$

where h_i and h' are the appropriate Hurewicz homomorphisms. If n is odd, and $n \neq 3$, we have from (*)

$$\begin{tikzcd} \pi_n(G/T) \arrow[r] \arrow[d] & \pi_n(G/T, H/T) \arrow[r, "d"] \arrow[d, "h'"] & \pi_{n-1}(H/T) \arrow[d, "0"] \\ 0 \arrow[r] & H_n(G/T, H/T) \arrow[r, "\partial"] & H_{n-1}(H/T) \end{tikzcd}$$

Thus $h' = 0$. From the commutative diagram

$$\begin{tikzcd} \pi_n(G/T, H/T) \arrow[r, "h'"] \arrow[d, "p_\sharp", "\cong"'] & H_n(G/T, H/T) \arrow[d, "p_*"] \\ \pi_n(G/H, *) \arrow[r, "h"] & H_n(G/H, *) \end{tikzcd}$$

we have that h is trivial.

For $n = 3$ we can easily show that $h' = 0$, hence $h = 0$. This follows from the fact that the inclusion induces a monomorphism $i_\sharp: \pi_2(H/T) \to \pi_2(G/T)$ (see below). Then exactness implies that $d: \pi_3(G/T, H/T) \to \pi_2(H/T)$ is trivial. Hence we have the same situation as for any odd n except that now $d = 0$ instead of $h_1 = 0: \pi_2(H/T) \to H_2(H/T)$. Hence h is trivial.

To see that $i_\sharp: \pi_2(H/T) \to \pi_2(G/T)$ is a monomorphism, we observe that the inclusion map $H \subset G$ is a fibre preserving map from the fibration $H \to H/T$ to the fibration $G \to G/T$. Thus we get the exact ladder

$$\begin{tikzcd} \dots \arrow[r] & \pi_2(H) \arrow[r] \arrow[d] & \pi_2(H/T) \arrow[r] \arrow[d, "i_\sharp"] & \pi_1(T) \arrow[r] \arrow[d, "\cong"] & \dots \\ \dots \arrow[r] & \pi_2(G) \arrow[r] & \pi_2(G/T) \arrow[r] & \pi_1(T) \arrow[r] & \dots \end{tikzcd}$$

Since $\pi_2(H) = \pi_2(G) = 0$ (because π_2 of a Lie group is zero), examination of the diagram reveals that $i_\sharp$ is a monomorphism.

The above theorem seems to be new; at any rate I could not find it in the literature. There are examples of homogeneous spaces as in the theorem with non-zero odd homology groups. One such space is G_2/SO_4. Thus the Hurewicz homomorphism is not trivially null. We easily obtain the following corollaries.

COROLLARY 5·4. *Let H be a closed connected subgroup in a compact, connected Lie group G. Let $\pi_n(G/H)$ be the first non-zero homotopy group. Then, if $\chi(G/H) \neq 0$*

(*a*) *n is even,*

(*b*) *$H_{n+1}(G/H; Z) = 0$.*

Proof. The Hopf–Samelson theorem (4) says $\chi(G/H) \neq 0$ if and only if H has maximal rank. Now apply the Hurewicz isomorphism theorem to Theorem 5·3.

COROLLARY 5·5. $H_3(G/H; Z) = 0$.

Proof. $\pi_1(G/H)$ is Abelian, hence G/H is simply connected (as is well known) by the above corollary. Thus n is at least 2 and hence $H_3(G/H; Z) = 0$.

These corollaries are easy consequences of a spectral sequence argument using the Bott–Borel theorem, the Hopf–Samelson theorem, and the fact that the odd homology groups of G/H are torsion. We may also prove the corollaries directly by using

Proposition 5·1 (*b*) and Lemma 4·1 and the Hurewicz isomorphism theorem. We do not need the Bott–Borel or the Hopf–Samelson theorems.

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