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Witnesses, Transgressions, and the Evaluation Map

DANIEL HENRY GOTTLIEB

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§1. Introduction. We study the relationship of the evaluation map $\omega: X^X \to X$ with the various transgression homomorphisms arising from fibrations with fibre X. We observe that the transgression factors through homomorphisms induced by ω. Thus (1) we may use facts about the evaluation map to calculate the transgression in some cases, or conversely (2) we may apply information about the transgression to discover facts about ω. As examples of this technique, using (1) we calculate homology with $\mathbf{Z}_p$ coefficients of the total space of any oriented fibration with fibre CP^n where p does not divide $n + 1$, and using (2) we calculate ω^* on integral cohomology where $X = CP^n$.

Next we consider the following question. Given a principal fibration $E \to B$, what can be said about the homology of the space of principal bundle maps, $L^*(E, E)$? We show this question is intimately related to ω^* in cohomology. Then, applying results on ω^* developed earlier, we compute the rational homology of $L^*(E, E)$ in terms of the homology of the space of self homotopy equivalences of B in the case of S^1 principal bundles over suitable base spaces. For example, we may calculate the rational homology of the space of equivariant maps with respect to that action of S^1 on S^3 which results in the Hopf fibration.

Finally, we use our study of $L^*(E, E)$ to show that $2\chi(M)\omega^* = 0$ for M a closed manifold and ω the evaluation map from the group of homeomorphisms of M to M.

In this paper we shall always assume that every space X has a base point $*$, but maps do not preserve base points unless it is specifically mentioned so. If M is a space of functions from $X \to Y$, then $\hat{\omega}: M \times X \to Y$ will always denote the evaluation map given by $\hat{\omega}(f, x) = f(x)$. We shall call $\hat{\omega}$ the *generalized evaluation map* or else an *action of* M *on* X when $X = Y$. Also $\omega: M \to Y$ will always denote evaluation at the base point; that is $\omega(f) = f(*)$. We shall always call ω the *evaluation map*. If an integer is denoted by p, then it is prime.

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The term witnesses, which appears in the title, will be defined later. It refers to a concept which gives $\hat{\omega}^*$ a geometric interpretation, an interpretation which will be needed in our investigation of $L^*(E, E)$.

I would like to thank James Becker and Reinhard Schultz for some helpful conversations.

§2. **Evaluation subgroups.** In this section we use Weingram's theorem to establish some results about ω_* on homotopy groups. These results will relate the transgression in the homotopy exact sequence of a fibration to the Hurewicz homomorphism.

Recall the n^{th} evaluation subgroup of a space X, written $G_n(X)$, is the image of $\omega_*: \pi_n(X^X; 1_X) \to \pi_n(X; *)$ where $\omega: X^X \to X$ is the evaluation map. Let $h: \pi_*(X) \to H_*(X; \mathbf{Z})$ be the Hurewicz homomorphism. We say X is *finitely co-connected if* $H_k(X; \mathbf{Z}) = 0$ for all $k \geqq N$ for some fixed N.

Theorem 1. *Let X be a finitely co-connected CW complex with $H_{2n}(X)$ finitely generated. Then $G_{2n}(X) \subset$ kernel of h.*

Corollary 2. *Let $X \to E \to B$ be a Hurewicz fibration with X as above. Let $d: \pi_{i+1}(B) \to \pi_i(F)$ be the transgression homomorphism in the homotopy exact sequence of the fibration. Then the composition*

$$\pi_{2n+1}(B) \xrightarrow{d} \pi_{2n}(F) \xrightarrow{h} H_{2n}(X; \mathbf{Z})$$

is trivial.

Corollary 3. *Let X be a finite CW complex with $\pi_2(X)$ finitely generated. Then $G_2(X) = 0$. If in addition $X \to E \to B$ is a Hurewicz fibration, then $d: \pi_3(B) \to \pi_2(X)$ is trivial.*

Proof of theorem 1. For any space X there exists a universal Hurewicz fibration $X \to E_\infty \to B_\infty$ with fibre the homotopy type of X, [1], or [5]. Now the transgression homomorphism $d_\infty: \pi_{i+1}(B_\infty) \to \pi_i(X)$ is related to $G_i(X)$ by $d_\infty(\pi_{i+1}(B_\infty)) = G_i(X)$. See theorem 2, §4 of [7].

Let $\alpha \in G_{2n}(X)$. We want to show that $h(\alpha) = 0$. We know there exists an $\alpha' \in \pi_{2n+1}(B_\infty)$ such that $d_\infty(\alpha') = \alpha$. Let $f: S^{2n+1} \to B_\infty$ be a map which represents α'. Then f induces the fibration $X \to f^*(E_\infty) \to S^{2n+1}$. Thus we have a map $g: \Omega S^{2n+1} \to X$ which arises from the fibration. In addition, α is in the image of

$$g_*: \pi_{2n}(\Omega S^{2n+1}) \to \pi_{2n}(X).$$

(This follows since g_* is essentially the same as $d: \pi_{2n+1}(S^{2n+1}) \to \pi_{2n}(X)$.)

Now assume $h(\alpha) \neq 0$. Then $g_*: H_{2n}(\Omega S^{2n+1}; \mathbf{Z}) \to H_{2n}(X; \mathbf{Z})$ is nontrivial. Now Weingram's theorem, Theorem 1.10 of [14], states: *Let $g: \Omega S^{2n+1} \to X$ be any map such that $g_*: H_{2n}(\Omega S^{2n+1}; \mathbf{Z}) \to H_{2n}(X; \mathbf{Z})$ is nontrivial. Assume $H_{2n}(X; \mathbf{Z})$ is finitely generated. Then X is not finitely co-connected.* This is precisely our situation, so theorem 1 is proved since we assumed X was finitely co-connected and so our assumption that $h(\alpha) \neq 0$ leads to a contradiction.

Proof of corollary 2. By [7], the image of $d:\pi_{i+1}(B) \to \pi_i(X)$ is contained in $G_i(X)$. Apply theorem 1.

Proof of corollary 3. Since X is a finite CW complex, the universal covering space $\tilde{X}$ is a finitely co-connected complex. Also $H_2(\tilde{X}; \mathbf{Z})$ is finitely generated since $\pi_2(X)$ is. Finally $G_i(\tilde{X}) \subset G_i(X)$ (here we identify $\pi_i(\tilde{X})$ with $\pi_i(X)$) for $i > 1$. See Theorem 6.2 of [8]. Thus, by theorem 1, $G_2(\tilde{X}) \subset \ker h = 0$. So $G_2(X) = 0$.

Remarks. a) Note that if $G_i(X) = 0$, then every Hurewicz fibration over S^{i+1} with X as the fibre has a cross-section. In particular, every fibration over S^3 with fibre a finite dimensional CW complex X such that $\pi_2(X)$ is finitely generated has a cross-section. This is Corollary 3.4 of [14]. b) If X is an H-space, then $\pi_i(X) = G_i(X)$. Let X be finitely co-connected with $H_*(X; \mathbf{Z})$ finitely generated. Then the Hurewicz homomorphism is trivial in even dimensions. This is Corollary 2.2 of [14].

§3. The evaluation map and the Serre spectral sequence. In this section we show that the transgression which arises in the Serre spectral sequence factors through ω^*, the homomorphism induced by ω. We then combine this fact with a theorem about ω^* to gain information concerning the Serre exact sequence.

Let G be a group of self-homotopy equivalences of a space F. We shall assume that G is connected so that all fibrations considered will be orientable. Let $G \to E \to B$ be a principal fibration. Then we have the commutative diagram

$$(1)\qquad \begin{array}{ccc} G \times F & \xrightarrow{\hat{\omega}} & F \\ \downarrow & & \downarrow \\ E \times F & \xrightarrow{\hat{\phi}} & \bar{E} \\ \downarrow & & \downarrow \\ B & \xrightarrow{1} & B \end{array}$$

where $\bar{E} = E \times_G F$ and $\hat{\phi}(e, x) = \langle e, x \rangle$. (We sometimes let G be the monoid of self-homotopy equivalences of F homotopic to 1_F. In this case we get a diagram similar to (1) for any orientable fibration $F \to \bar{E} \to B$. In this case, E will be the space $\bar{E}^{(F)}$ consisting of maps from F into fibres of $\bar{E}$ such that each map is a homotopy equivalence of F onto the fibre. Also $\hat{\phi}$ will be the generalized evaluation map.)

Diagram (1) gives rise to the commutative diagram

$$(2) \qquad \begin{array}{ccc} G & \xrightarrow{\omega} & F \\ \downarrow & & \downarrow \\ E & \xrightarrow{\phi} & \bar{E} \\ \downarrow & & \downarrow \\ B & \xrightarrow{1} & B \end{array}$$

by evaluation at the base point.

Diagrams (1) and (2) gives rise to mappings of the Serre exact sequences associated to the fibrations in question. We shall always let G be connected, so that the Serre spectral sequences involved do not require local coefficients. From diagram (2), we obtain the following theorem.

Theorem 4. *$F \to \bar{E} \to B$ and G as above. Let F be m-connected and B be n-connected. Then the transgression $\tau : H^i(F; \pi) \to H^{i+1}(B; \pi)$ is defined for $i \leqq m + n + 1$ and the following diagram commutes for $i \leqq 2n$ when τ is defined:*

$$\begin{array}{ccc} H^i(F;\pi) & \xrightarrow{\tau} & H^{i+1}(B;\pi) \\ & \searrow^{\omega^*} \quad \nearrow_{\bar{\tau}} & \\ & (\text{image of } \omega^*) \subset H^i(G;\pi) & \end{array}$$

where $\bar{\tau}$ is the transgression defined on the appropriate subgroup and π is a field. If π is an arbitrary group, then the diagram commutes for $i = n$ and $i = n + 1$ and $i = 2$.

Proof. Let $x \in H^i(F; \pi)$. If x survives to $E_r^{0,i}$, let $k_r(x) \in E_r^{0,i}$ denote the element represented by x. From diagram (2), we have a homomorphism, $\{\phi_r^{p,q}\} : E_r^{p,q} \to \tilde{E}_r^{p,q}$, of spectral sequences where $\{E_r^{p,q}\}$ and $\{\tilde{E}_r^{p,q}\}$ represent the spectral sequences corresponding to the fibrations $F \to \bar{E} \to B$ and $G \to E \to B$ respectively.

If an element $x \in H^i(F; \pi)$ transgresses, then by the naturality of transgressions we know that $\omega^*(x) \in H^i(G; \pi)$ transgresses. In the range where $i \leqq m + n$, every element in $H^i(F; \pi)$ transgresses. Hence every element in the image of ω^* must transgress. Thus $\bar{\tau}(\omega^*(x)) \in \tilde{E}_i^{i+1,0} =$ quotient of $H^{i+1}(B; \pi)$. To prove the theorem, we must show that $\tilde{E}_i^{i+1,0}$ is actually equal to $H^{i+1}(B)$ in the ranges given by the hypothesis. Then the fact that $\phi_2^{i,0}$ is the identity will yield the theorem.

We have $d_r : \tilde{E}_r^{i-r,r} \to \tilde{E}_r^{i+1,0}$. When $i = n$ or $n + 1$ we see that $\tilde{E}_r^{i-r,r} = 0$ for $i > r > 1$. Thus $E_i^{i+1,0} = H^{i+1}(B; \pi)$.

Now assume that π is a field. Then $\tilde{E}_2^{p,q} = H^p(B; \pi) \otimes H^q(G; \pi)$. Let $i < 2n + 1$. As before, we must show that $d_r : \tilde{E}_r^{i-r,r} \to \tilde{E}_r^{i+1,0}$ is zero for $r < i$. For $i - r \leqq n$ we have $\tilde{E}_r^{i-r,r} = 0$. So we must show that $d_r = 0$ for $i - r \geqq n + 1$. Let $x \in \tilde{E}_2^{i-r,r} \cong H^{i-r}(B) \otimes H^r(G)$. Then x is the sum of terms of the form $a \otimes b$ where $a \in H^{i-r}(B)$ and $b \in H^r(G)$. Now $d_r(k_r(a)) = 0$ for all r. Also $d_r(k_r(b)) = 0$ for all $r < n$ since B is n-connected. Thus $d_r(k_r(a) \otimes k_r(b)) = 0$ for $r < n$ since d_r is a derivation. Thus $d_r = 0$ for $r < n$. Hence we have $E_i^{i+1,0} \cong H^{i+1}(B)$ for $i < 2n + 1$.

The theorem is true for $i = 2$ since $d_2:\tilde{E}_2^{1,1} \to \tilde{E}_2^{3,0}$ is trivial because $G \to E \to B$ is orientable.

We can combine theorem 4 with results about the evaluation map in [10] to obtain results about fibrations. For example, consider the following result from [10] dualized to cohomology.

Let $k \in H^i(F; \mathbf{Z}_p)$. We shall say that k is *primitive under the action* $\hat{\omega}:G \times F \to F$ if $\hat{\omega}^*(k) = (\omega^*(k) \otimes 1) + (1 \otimes k)$ and $\omega^*(k) \neq 0$. For example, if F is $(m-1)$-connected and $k \in H^*(F; \mathbf{Z}_p)$ has dimension less than $2m$, then k is primitive under any action if and only if $\omega^*(k) \neq 0$. Let $[k]_p$ denote the truncated polynomial ring generated by k with height p (that is, $[k]_p$ is generated by $1, \cdots, k^{p-1}$).

Theorem 5. ([10], *Theorem* 2 *dualized to cohomology.*) *Let $k \in H^i(F; \mathbf{Z}_p)$ be primitive under some action. Then there exists some $\mathbf{Z}_p$ module $M \subset H^*(F; \mathbf{Z}_p)$ such that*

$$H^*(F; Z_p) \cong [k]_p \otimes M \text{ as } \mathbf{Z}_p\text{-modules if } i \text{ is even,}$$

and

$$H^*(F; \mathbf{Z}_p) \cong [k]_2 \otimes M \text{ as } \mathbf{Z}_p\text{-modules if } i \text{ is odd.}$$

In this theorem p is a prime or $p = \infty$.

Now combining theorem 4 and theorem 5 with the Serre exact sequence

$$\cdots \to H^i(B) \xrightarrow{p^*} H^i(E) \xrightarrow{i^*} H^i(F) \xrightarrow{\tau} H^{i+1}(B) \to \cdots$$

for $i \leqq m + n$, we may obtain results like the following. Recall B is n-connected and F is m-connected.

Corollary 6. *$p^*:H^i(B; R) \to H^i(E; R)$ is injective if $i \leqq 2n$ and $i \leqq m + n$ and if F is a finite complex such that $\chi(F) \neq 0$. Here $R = \mathbf{Z}_\infty$, the rationals.*

Proof. It follows easily from theorem 5, when $p = \infty$, and from the proof of theorem 3 of [10] that $\omega^*:H^*(F; R) \to H^*(G; R)$ is trivial. Thus by theorem 4, $\tau:H^i(F; R) \to H^{i+1}(B; R)$ is trivial when $i \leqq 2n$ and $i \leqq n + m$. Now the exact sequence yields the result.

Corollary 7. *Let $CP^n \to E \to B$ be an oriented fibration. Then if $n + 1 \not\equiv 0 \pmod{p}$, we have $H(E; \mathbf{Z}_p) \cong H^*(B; \mathbf{Z}_p) \otimes H^*(CP^n; \mathbf{Z}_p)$ as vector spaces.*

Proof. Let $\alpha \in H^2(CP^n; \mathbf{Z}_p)$ be a generator. If $\omega^*(\alpha) \neq 0$, then p must divide $\chi(CP^n) = n + 1$ by theorem 5. Hence $\omega^*(\alpha) = 0$. Hence α is in the image of i^*. Hence $H^*(CP^n; \mathbf{Z}_p)$ is in the image of i^*. Hence the Serre spectral sequence collapses.

In the following theorem we use theorem 4 to compute ω^* for CP^n. Let L denote the space of self-homotopy equivalences on CP^n homotopic to the identity and let L_0 be the base-point-preserving maps in L. We give L the compact-open topology.

Theorem 8. *Let $\alpha \in H^2(CP^n; \mathbf{Z}) \cong \mathbf{Z}$ be a generator. Then $\omega^*(\alpha)$ is an element of order $n+1$ which generates $H^2(L; \mathbf{Z}) \cong \mathbf{Z}_{n+1}$.*

Proof. It is immediate from the Federer spectral sequence that $\pi_1(L_0) \cong \mathbf{Z}$ and $\pi_2(L)$ is finite. On the other hand, $\pi_1(L) \cong \mathbf{Z}_{n+1}$ follows from obstruction theory; see Theorem 11 on page 452 of Spanier [13].

Consider the universal fibration $CP^n \to E_\infty \to B_\infty$. Then for $i > 1$, $\pi_i(B_\infty) \cong \pi_{i-1}(L)$, and $\pi_i(E_\infty) \cong \pi_{i-1}(L_0)$ since E_∞ is the classifying space for L_0, [11]. Let $\tilde{B}_\infty$ be the universal covering space of B_∞. Then we obtain a fibration $CP^n \to \tilde{E}_\infty \to \tilde{B}_\infty$ induced from the universal fibration by the covering projection. Note that $\tilde{E}_\infty$ is the universal covering space of E_∞. The homotopy exact sequence gives us

$$\pi_3(\tilde{B}_\infty) \xrightarrow{d} \pi_2(CP^n) \xrightarrow{i_*} \pi_2(\tilde{E}_\infty) \xrightarrow{p_*} \pi_2(\tilde{B}_\infty) \to 0$$

which becomes

$$\pi_3(\tilde{B}_\infty) \xrightarrow{d} Z \xrightarrow{i_*} Z \xrightarrow{p_*} Z_{n+1} \to 0$$

so i_* is multiplication by $n+1$.

Consider the commutative diagram

$$\begin{array}{ccc} \pi_2(CP^n) & \xrightarrow{i_*} & \pi_2(\tilde{E}_\infty) \\ \cong \big\downarrow h & & \cong \big\downarrow h \\ H_2(CP^n) & \xrightarrow{i_*} & H_2(\tilde{E}_\infty). \end{array}$$

Then i_* is multiplication by $n+1$ on homology. Hence i^* is multiplication by $n+1$ on cohomology.

By the universal coefficient theorem, $H^3(\tilde{B}_\infty; \mathbf{Z}) \cong Z_{n+1} \oplus F$ where F is a free abelian group. In fact $F \cong 0$ by the Hurewicz isomorphism theorem and the universal coefficient theorem since $\tilde{B}_\infty$ is simply connected and $\pi_3(\tilde{B}_\infty) \cong \pi_2(L)$ is a finite group.

We also have $H^2(L; \mathbf{Z}) \cong \mathbf{Z}_{n+1}$ as follows: We know that $H^2(L; \mathbf{Z}) \cong \mathbf{Z}_{n+1} \oplus F$ where F is the free group of rank b_2, the second Betti number of $H_2(L; \mathbf{Z})$; but $H_2(L; \mathbf{Z})$ is finite since $\pi_2(L)$ is finite implies that $\pi_3(\tilde{B}_\infty)$ is finite implies that $H_3(\tilde{B}_\infty; \mathbf{Z})$ is finite. Then an easy argument with the Serre spectral sequence for the universal fibration $L \to \bar{E} \to \tilde{B}_\infty$, where $\bar{E}$ is essentially contractible, shows $H_2(L; \mathbf{Z})$ is finite, so $H^2(L; \mathbf{Z}) \cong \mathbf{Z}_{n+1}$.

Now we are in a position to use theorem 4 to calculate $\omega^*: H^2(CP^n; \mathbf{Z}) \cong \mathbf{Z} \to \mathbf{Z}_{n+1} \cong H^2(L; \mathbf{Z})$. From the Serre exact sequence we have

$$\to H^2(\tilde{E}_\infty) \xrightarrow{i^*} H^2(CP^n) \xrightarrow{\tau} H^3(\tilde{B}_\infty)$$

which is

$$\to \mathbf{Z} \xrightarrow{(n+1)} \mathbf{Z} \xrightarrow{\tau} \mathbf{Z}_{n+1} .$$

Thus τ maps $\mathbf{Z}$ onto $\mathbf{Z}_{n+1}$. But $\omega^*: H^2(CP^n) \to H^2(L) \cong \mathbf{Z}_{n+1}$ is a factor of τ, so ω^* must be onto, which proves the theorem.

Reinhard Schultz has another method for proving theorem 8.

§4. Witnesses. Ronald Brown, [3], has shown that there is an isomorphism $\Theta: \sum_{i+j=n} H^i(X; H^j(Y; G)) \to H^n(X \times Y; G)$ which is natural with respect to maps of the form $f \times 1: X \times Y \to X' \times Y$. This fact may appear to follow immediately from the Kunneth formula or the Serre spectral sequence but I don't believe it does. Note the equally plausible statement that Θ is natural with respect to maps of the form $1 \times f: X \times Y \to X \times Y'$ is false, as is shown by Brown in [3].

In this section we shall observe that Brown's result may be proved by using an idea of J. P. Meyer [12]. We shall use the splitting given by Brown's theorem to write $\hat{\omega}^*(k)$ as a sum of terms $\omega_0(k) + \omega_1(k) + \cdots + \omega_n(k)$ called *witnesses of* $k \in H^n(Y; G)$ *in* M, a space of self homotopy equivalences of Y. These witnesses will then determine the homotopy class of the map $k_\#: M \to K(G, n)^Y$ where $k_\#$ denotes composition by $k: Y \to K(G, n)$. In the next section we shall show that $k_\#$ determines spaces of bundle maps up to homotopy type.

Let G be an abelian group. Then $K(G, n)$ can be thought of as a topological abelian group, [4]. Thus $K(G, n)^Y$, space of maps of Y into $K(G, n)$ can be given a topological abelian group structure in a natural way. Then by [4], we know that $K(G, n)^Y$ is homotopy equivalent, *by an h-homomorphism*, to

$$\prod_{i=0}^{n} K(H^{n-i}(Y; G), i)$$

with the group structure defined by the product. Let ι_i be the fundamental class of $K(H^{n-i}(Y; G), i)$. We may regard ι_i as an element of $H^i(K(G, n)^Y; H^{n-i}(Y; G))$.

Now we define the isomorphism $\Theta: \sum_i H^i(X; H^{n-i}(Y; G)) \to H^n(X \times Y; G)$ as follows: Every $\omega \in \sum_i H^i(X; H^{n-i}(Y; G))$ may be thought of as a tuple $(\omega_0, \cdots, \omega_n)$. This tuple gives rise to a map $f: X \to K(G, n)^Y$ such that $\omega_i = f^*(\iota_i)$ for all i. Let $\hat{f}$ be the adjoint map $\hat{f}: X \times Y \to K(G, n)$. Then $\Theta(\omega) = \hat{f}^*(\iota)$ where ι is the fundamental class of $K(G, n)$.

It is immediate from the definition that Θ is bijective and natural with respect to maps of the form $f \times 1: X \times Y \to X' \times Y$. That Θ is a homomorphism follows from the fact that $K(G, n)^Y$ is h-homotopy equivalent to $\prod_i (K(H^{n-i}(Y; G), i))$.

Let M be a space of self-homotopy equivalences of Y. Let $k \in H^n(Y; G)$. Then k may be regarded as a map $k: Y \to K(G, n)$. Let $k_\#: M \to K(G, n)^Y$ be the map given by $f \to k \circ f$. Consider the commutative diagram

$$\begin{array}{ccc} M \times Y & \xrightarrow{k_\# \times 1} & K(G, n)^Y \times Y \\ \downarrow \hat{\omega} & & \downarrow \hat{\omega} \\ Y & \xrightarrow{k} & K(G, n). \end{array}$$

We see that $\hat{\omega}^*(k) = \omega_0(k) + \cdots + \omega_n(k)$ where $\omega_i(k)$ is the component of $\omega^*(k)$ in $H^i(M; H^{n-i}(Y; G))$ under the direct sum decomposition of $H^n(M \times Y; G)$ given by Θ. We call $\omega_i(k)$ the ith *witness of k in M*. The set of witnesses of k determines $k_\#$ up to homotopy since $\omega_i(k) = k_\#^*(\iota_i)$. If $i: M' \to M$ is a map of spaces of self homotopy equivalences of Y where $i(f) = f$ as functions, then the witnesses of k in M pull back to the witnesses of k in M'.

Note that in situations where it makes sense, the 0th witness $\omega_0(k) = 1 \times k$ and the nth witness $\omega_n(k) = \omega^*(k) \times 1$.

§5. *Bundle map theory.* We summarize the main points of bundle map theory. See [9] for details. Let $G \to E \to B$ and $G \to E' \to B'$ be principal bundles and let $\tilde{f}: E \to E'$ be a principal bundle map. Then we have a Serre fibration

$$L^{**}(E, E') \to L^*(E, E') \xrightarrow{\Phi} L(B, B)$$

where $\Phi(\tilde{f})$ is the induced map $\phi(\tilde{f}): B \to B'$, and $L(B, B')$ is the space of maps from $B \to B'$ homotopic to $\Phi(\tilde{f})$, and $L^*(E, E')$ is the space of principal bundle maps from $E \to E'$ inducing maps in $L(B, B')$. We assume that the mapping spaces have the compact-open topology. Also $L^{**}(E, E')$ is the space of principal bundle maps covering $\Phi(\tilde{f}): B \to B'$. Now $L^{**}(E, E')$ is homeomorphic to $L^{**}(E, E)$, the space of bundle equivalences from $E \to E$. If $G \to E_G \to B_G$ is the universal bundle, then $L^*(E, E_G)$ is essentially contractible, so $L^{**} \to L^*(E, E_G) \to L(B, B_G)$ is a universal principal fibration. Now if $k: B \to B_G$, we have

$$\begin{array}{ccc} L^{**} & \xrightarrow{\cong} & L^{**} \\ \downarrow & & \downarrow \\ L^*(E, E) & \xrightarrow{\hat{k}_\#} & L^*(E, E_G) \\ \downarrow \Phi & & \downarrow \Phi \\ L(B, B) & \xrightarrow{k_\#} & L(B, B_G) \end{array}$$

and the first column is the pullback of the second by the composition map $k_\#$ induced by k. Thus if we know the homotopy class of $k_\#$, we may be in a position to calculate $L^*(E, E)$.

Now let M be a space of functions from $B \to B'$. We extend the discussion above for M. Assume we have a map $i: M \to L(B, B')$ such that $f = i(f)$ as maps.

The pullback of i serves to define M^* and M^{**} in the diagram

$$\begin{array}{ccc} M^{**} & \longrightarrow & L^{**}(E, E') \\ \downarrow & & \downarrow \\ M^* & \xrightarrow{\tilde{i}} & L^*(E, E') \\ \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \Phi} \\ M & \xrightarrow{i} & L(B, B') \end{array}$$

Again M^{**} is homeomorphic to $L^{**}(E, E)$. In addition note that $k_\# = i \circ k_\#$ (recall $k_\#$ is composition on the left by k). Thus if we know the homotopy class of $k_\#: M \to L(B, B_G)$, we know in principle the homotopy type of M^*.

Let $G = K(\pi, n-1)$. Then $B_G = K(\pi, n)$ and $L(B, K(\pi, n)) = \prod_{i=1}^n K(\pi_i, i)$ where $\pi_i = H^{n-i}(B; \pi)$. Thus, as we have seen, the homotopy type of $k_\#$ is determined by its set of witnesses $(\omega_0, \cdots, \omega_n) \in \sum_{i=1}^n H^i(L(B, B); H^{n-i}(B; \pi))$.

We shall look at some examples: Let $G \cong \pi \cong K(\pi, 0)$. Suppose $p: \tilde{B} \to B$ is a regular covering of B where $\pi_1(B)/\pi_1(\tilde{B}) \cong \pi$. Then it is induced by a map $k: B \to K(\pi, 1)$. The space $L^*(\tilde{B}, \tilde{B})$ is determined by the homotopy class of $k_\#$. If π is abelian, $k_\#$ is determined by the witness $\omega_1 = \omega^*(k)$. If $\omega^*(k) = 0$, then $k_\#$ is homotopic to a constant map and $L^*(\tilde{B}, \tilde{B})$ is homotopy equivalent to $L(B, B) \times \pi$. In that case we have the commutative diagram

$$(*) \qquad \begin{array}{ccc} L^*(\tilde{B}, \tilde{B}) & \xrightarrow{\omega} & \tilde{B} \\ {\scriptstyle s}\uparrow\downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle p} \\ L(B, B) & \xrightarrow{\omega} & B \end{array}$$

where $s: L \to L^*$ is a cross-section. Thus $\omega: L(B, B) \to B$ factors through $p \circ \omega: L^*(\tilde{B}, \tilde{B}) \to B$. We may apply this fact to get the following result.

Theorem 9. *Let Y be a closed topological manifold and M be a space of homeomorphisms on Y with topology as in the third paragraph above. Then*

$$2\chi(Y)\omega^*: \tilde{H}^*(Y; R) \to \tilde{H}^*(M; R)$$

is trivial for any ring with unit R as coefficients.

Proof. Let $w_1 \in H^1(Y; \mathbf{Z}_2)$ be the Stiefel–Whitney class of Y in the sense of Fadell. Then $\omega^*(w_1) = 0$, see [9], §8. Thus from Diagram (*) we have (where $\tilde{Y}$ is oriented covering of Y)

$$\begin{array}{ccc} M^* & \xrightarrow{\omega} & \tilde{Y} \\ {\scriptstyle s}\uparrow\downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle p} \\ M & \xrightarrow{\omega} & Y \end{array}$$

Now we know that $\chi(\tilde{Y})\omega^*: \tilde{H}^*(\tilde{Y}; R) \to H^*(M^*; R)$ is the zero map (see theorem (8.13) of [9]). The diagram yields the theorem since $\chi(\tilde{Y}) = 2\chi(Y)$.

Remark. For $R = \mathbf{Z}_2$, we have $\chi(M)\omega^* = 0$ since every closed manifold is $\mathbf{Z}_2$-orientable. Also a much more general version of the above theorem is true.

In the case of principal S^1-bundles, we may determine L^* as follows. The S^1-bundle $E \to B$ is classified by a map $k: B \to B_{S^1} = K(\mathbf{Z}, 2)$. Thus $k_\#$ is determined by the 1st and 2nd witnesses. We may frequently determine these witnesses. Thus if $\pi_1(B)$ has no elements of infinite order, then $H^1(B; \mathbf{Z}) = 0$ so $\omega_1(k) = 0$. This proves

Lemma 10. *The first witness of $k \in H^2(B; Z)$ is zero if $\pi_1(B)$ is finite. Hence L^{**} is homotopy equivalent to S^1.*

If $G_1(B)$ is trivial ($G_1(B)$ is the image of $\omega_*: \pi_1(B^B; 1_B) \to \pi_1(B)$), then $L^*(\tilde{B}, \tilde{B})$ is the union of disjoint copies of $L(B, B)$. Here $\tilde{B}$ is the universal covering space of B. This leads to a commutative diagram

$$\begin{array}{ccc} L^*(\tilde{B}, \tilde{B}) & \xrightarrow{\ \omega\ } & \tilde{B} \\ s\uparrow\downarrow\Phi & & \downarrow p \\ L(B, B) & \xrightarrow{\ \omega\ } & B \end{array}$$

where s is the cross-section into the identity component of $L^*(\tilde{B}, \tilde{B})$. Thus $\omega = p\omega s$.

Since $\pi_1(\tilde{B}) = 0$, we know that $\omega^*: H^2(\tilde{B}; \mathbf{Q}) \to H^2(L^*; \mathbf{Q})$ must be trivial if $\tilde{B}$ is finite dimensional. Here $\mathbf{Q}$ is the rational numbers. (See theorem 5). Thus $\omega^*: H^2(B; \mathbf{Q}) \to H^2(L(B, B); \mathbf{Q})$ is trivial. Hence the image of ω^* in $H^2(B; \mathbf{Z})$ consists of torsion elements. This proves

Lemma 11. *If B is finite dimensional and $G_1(B) = 0$, then*

$$\omega_2(k) \in H^2(L(B, B); \mathbf{Z}),$$

the second witness for $k \in H^2(B; \mathbf{Z})$ has finite order.

A consequence of these lemmas is the following theorem. We say E *splits rationally* as a product of A and B if $H^*(E; \mathbf{Q}) \cong H^*(A; \mathbf{Q}) \otimes H^*(B; \mathbf{Q})$ as groups.

Theorem 12. *Let B be a finite polyhedron and $E \to B$ a principal S^1-bundle. Suppose $\chi(B) \neq 0$ and $\pi_1(B)$ is finite. Then $L^*(E, E)$ splits rationally as a product of $L(B, B)$ and S^1.*

Proof. We need to show that for $k \in H^2(B; \mathbf{Z})$, which classifies $E \to B$, there is an integer m such that $(mk)_\#$ is homotopy trivial. Thus we must show that the witnesses for k have finite orders. Now $\omega_1 = 0$ by lemma 10 and ω_2 has finite order by lemma 11 and the fact that $G_1(B) = 0$. (Theorem (IV.1), [6]). Note that $L^{**}(E, E)$ is actually homotopy equivalent to S^1.

An easy spectral sequence argument, similar to the ones in §3, shows that the spectral sequence for $S^1 \to L^* \to L$ collapses.

Remark. The theorem is true if $\chi(B) \neq 0$ is replaced by $G_1(B) = 0$ and if

$L(B, B)$ is replaced by any space of maps M from $B \to B$ such that there is a map $i: M \to L(B, B; 1_B)$ such that $i(f) = f$.

Theorem 13. *Let X be a compact connected CW complex. Then for a principal S^1-bundle over ΣX, the space $L^*(E, E)$ splits rationally, (and also splits mod p where p is an odd prime), as a product of $L(B, B)$ and S^1.*

Proof. Let k classify $S^1 \to E \to \Sigma X$. Then $\omega_1(k) \in H^1(\Sigma X; \mathbf{Z}) = 0$ so L^{**} is S^1 and $\omega_1 = 0$. Now $\omega_2 = \omega^*(k)$. We need to show that $\omega^*(k)$ is of order two.

Consider $\hat{\omega}^*(k) = 1 \times k + \omega^*(k) \times 1$. Now $k^2 = 0$ since cup products in a suspension are trivial. Thus $0 = \hat{\omega}^*(k^2) = 2\omega^*(k) \times k$. Now k must have infinite order since $H^2(\Sigma X; \mathbf{Z})$ is a free abelian group. Thus if k is a generator of $H^2(\Sigma X; \mathbf{Z})$, we must have $2\omega^*(k) = 0$. Thus $2\omega^*(k) = 0$ for arbitrary k.

As examples of the theory outlined above, we calculate the homotopy type of $L^*(E, E)$, where E is a principal S^1-bundle over RP^{2n} or over CP^n.

Example 1. For a principal S^1-bundle over RP^{2n}, we have $L^*(E, E)$ is homotopy equivalent to $L(RP^n, RP^n) \times S^1$.

Proof. Let $\alpha \in H^2(RP^{2n}; \mathbf{Z})$. The first witness $\omega_1(\alpha) = 0$ since $H^1(RP^{2n}; \mathbf{Z}) = 0$. Thus $L^{**}(E, E)$ is homotopy equivalent to S^1. Since $\chi(RP^{2n}) = 1$ is not zero, we have $G_1(RP^{2n}) = 0$. As in the proof of Lemma 11, we obtain a commutative diagram

$$\begin{array}{ccc} L^*(S^{2n}, S^{2n}) & \xrightarrow{\omega} & S^{2n} \\ s \uparrow \Big\downarrow \phi & & \Big\downarrow \\ L(RP^{2n}, RP^{2n}) & \xrightarrow{\omega} & RP^{2n} \end{array}$$

Thus $\omega_2(\alpha) = \omega^*(\alpha) = 0$. Hence $\alpha_\#$ is homotopic to a constant, so the fibration it induces, $S^1 \to L^*(E, E) \to L(RP^{2n}, RP^{2n})$, must be trivial.

Example 2. Let $\alpha \in H^2(CP^n; \mathbf{Z})$ be the generator. Then $\omega_1(\alpha) = 0$ and $\omega_2(\alpha) = \omega^*(\alpha)$ is the generator of $H^2(L(CP^n, CP^n; 1)) = \mathbf{Z}_{n+1}$ by theorem 8. Thus $L^*(E, E)$ is homotopy equivalent to the fibre of the map $L(CP^n, CP^n; 1) \to K(\mathbf{Z}, 2)$ induced by $\omega^*(\alpha)$, where $E \to CP^n$ is induced by α.

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Purdue University

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