THE TOTAL SPACE OF UNIVERSAL FIBRATIONS

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It is shown that the total space of a universal fibration for a fibre $F$ is a classifying space for the monoid of self homotopy equivalences of $F$ which fix the base point.

For any space $F$, there exists a universal Eurewicz fibration $F \rightarrow E_\infty \rightarrow B_\infty$, where $B_\infty$ is a CW complex which classifies Hurewicz fibrations over CW complexes (see Dold [2], Theorem 16.9.). Now $B_\infty$ is the Dold-Lashof classifying space for the monoid of self homotopy equivalence of $F$, which we shall denote by $F^\ast$. At least, this is the case when $F$ is a CW complex. (See Allaud [1], § IV.) The purpose of this note is to show that $E_\infty$ is the classifying space for the monoid of self equivalences of $F$ leaving the base point fixed, denoted $F^\ast_\infty$, when $F$ is a CW complex with homotopy equivalent path components. In fact, we shall show $E_\infty$ is the base space of a Serre fibration with fibre $F^\ast_\infty$ and a total space which is essentially contractible. We need this characterization of $E_\infty$ in order to calculate the induced homomorphism on integral cohomology of the evaluation map $\omega: X^2 \rightarrow X$ where $X = CP^\ast$. This is done in [3].

Let $D = p^\ast(E_\infty)$, the pullback of $E_\infty$ by $p: E_\infty \rightarrow B_\infty$. Thus $D = \{(e, e') \in E_\infty \times E_\infty \mid p(e) = p(e')\}$, and $\bar{p}: D \rightarrow E_\infty$ given by $\bar{p}(e, e') = e$ is the projection. Let $D^\ast_\infty$ be the set of maps of $F \rightarrow D$ endowed with the $C^0$ topology such that:

(a) Each map carries $F$ into some fibre of $D \rightarrow E_\infty$ and is a homotopy equivalence of $F$ and the fibre.
(b) Each map carries the base point, $\ast$, into a point of the form $(e, e)$.

Let $q: D^\ast_\infty \rightarrow E_\infty$ be given by $q(f) = \bar{p} \circ f(\ast)$.

**THEOREM.**

1. $q$ is a Serre Fibration, and if $F$ is locally compact $q$ is a Hurewicz fibration.
2. The fibre of $q$ is $F^\ast_\infty$.
3. There is a fibrewise action $D^\ast_\infty \times F^\ast_\infty \rightarrow D^\ast_\infty$ if $F$ is locally compact.
4. $D^\ast_\infty$ is essentially contractible.

**Proof of (1).** First note that $q$ is onto since all the components of $F$ have the same homotopy type.

We shall assume that $F$ has a whisker. That is, assume $F$ has
the form $J \vee F'$ where $J$ is the unit interval with the base point * being the 1 and with $0 \in F'$. Every space is homotopy equivalent to a space of this type, so we do not lose any generality.

Let $X$ be a compact polyhedron (or $F$ is locally compact). We must show that $p$ has the covering homotopy property with respect to any map $X \to D_f$. Since $X$ is compact (or $F$ is locally compact) $X \times F$ is a CW complex; so we may consider the adjoint map

$$f: X \times F \to D$$

where $f$ is a fibre preserving map which carries $X \times *$ into $\Delta \subset D$, where $\Delta = \{(e, e) \mid e \in E_w\}$. Then the covering homotopy property translates into a statement involving a fibre homotopy of $f$ which at each stage sends $X \times *$ into $\Delta$. Reflecting on the definition of $D$, we see that the covering homotopy property is equivalent to the following statement: Let $f: X \times F \to E_w$ be a fibre map and let $h: X \to E_w$ be any homotopy such that $h(x) = f(x, *)$ for all $x \in X$. Then there exists a fibre homotopy $h: X \times F \to E_w$ such that $\tilde{h}(x, o) = h(x)$ and $\tilde{h}(x, *) = f$.

Now this statement is a special case of the statement that $q$ has the covering homotopy extension property for the space $X \times F$ relative to $X \times *$. See Hu, page 62 [4] for the definition. But this follows from Satz 5.38, page 107 in [5]. (The fact that $*$ is on the end of a whisker allows us to satisfy the technical requirements of Satz 5.38 concerning a halo about $X \times *$.)

**Proof of (4).** First note that (2) and (3) are obviously true. The action in (3), $D_f \times F \to D_f$ is given by $(g, f) \to g \cdot f$. (This action is continuous if $F$ is locally compact.)

Now we shall show that $D_f$ is essentially contractible. That is, any map $X \to D_f$ is homotopy trivial if $X$ is a finite CW complex. Consider the adjoint map $g: X \times F \to D$. Then $g$ is defined by, and defines, a fibre map $h: X \times F \to E_w$ by means of the relation

$$g(x, y) = (h(x, *), h(x, y)) \in D.$$

Now $h$ can be extended to a fibre map $H: CX \times F \to E_w$, (because $E_w$ is essentially contractible, [1] Theorem 4.1). We define a fibre map $G: CX \times F \to D$ by

$$G(x, y) = (H(x, *), H(x, y)),$$

$x \in CX, y \in F$.

Note that $G$ extends $g$. The adjoint situation now shows that our original map $X \to D_f$ factors through $X \to CX \to D_f$.

**COROLLARY.**

$$\pi_1(E_w) \cong \pi_1^{-1}(F').$$

This corollary plays an important role in the computation of the homomorphisms induced in cohomology by the evaluation map $\omega: F' \to F$. The program is based on the use of the Federer spectral sequence and obstruction theory to compute some homotopy groups of $F'$, and hence of $E_w$ by the corollary. From the homotopy group information, we obtain information about the cohomology of $E_w$. Then we use the Serre exact sequence and the slogan "$\omega$ factors through the transgression" to recover information about $\omega$.

This program has been used successfully to compute $\omega$ for $H^k(CP^n; \mathbb{Z})$. See Theorem 16 of [3].

**REFERENCES**


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