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COVERINGS OF FIBRATIONS

by

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1. Introduction

Let $E \xrightarrow{p} B$ be a fibration. Then a covering map $\tilde{E} \xrightarrow{\pi} E$ gives rise to a fibration $\tilde{E} \xrightarrow{p\pi} B$. If the fibre of this fibration is connected, then the fibre is a covering space of the fibre of $E \to B$. In this case, we shall call $\tilde{E}$ *a covering fibration of the fibration* $F \to E \xrightarrow{p} B$ *extending the covering* $\pi : \tilde{F} \to F$.

This note takes up the following questions.

A. Given a fibration $F \to E \to B$ and a covering $\tilde{F} \xrightarrow{\pi} F$, when can we find a covering $\tilde{E}$ extending $\tilde{F}$?

B. Given a space F and a covering space $\tilde{F}$, under what conditions can we always find a covering fibration extending $\tilde{F}$ for any fibration with fibre F?

For $\tilde{F}$ a universal covering space, we can answer question A (see Theorem 1) in terms of conditions on the fundamental groups of the fibration $F \to E \to B$. For oriented fibrations and universal coverings $\tilde{F}$, we find the answer to B depends upon $G_1(F)$, (see Theorem 2). Some results on covering fibrations extending non-universal coverings $\tilde{F}$ are found in § 3.

The existence of covering fibrations leads to various applications. The most striking of them is theorem 15 which generalizes a theorem of Borel's [1], lemma 3.2.

By fibration, we shall mean Hurewicz fibration (i.e. $F \to E \xrightarrow{p} B$ has the homotopy covering property). We assume that E and F are path connected, locally path connected, and semi-locally 1-connected. Most of the results proved will obviously be true for other types of fibrations. In § 3, we consider fibre bundles (locally homeomorphic to a product of a neighborhood in the base and the fibre). By an oriented fibration, we shall mean the strongest possible interpretation. That is, $F \to E \to B$ is *oriented* if $\pi_1(B)$ operates trivially on $\xi(F)$, where $\xi(F)$ is the group of homotopy classes of self homotopy equivalences. By $\xi_0(F)$ we shall mean the group of based homotopy classes of based self homotopy equivalences.

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Also $\tilde{X}$ will always denote a covering space of X. Covering spaces are always assumed to be path connected.

2. Covering fibrations extending universal coverings

First we answer question A for universal coverings.

THEOREM 1: *Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration with connected fibre F and let $\tilde{F}$ be the universal cover of F. Then there exists a covering fibration $\tilde{E}$ of $E \xrightarrow{p} B$ extending $\tilde{F}$ if and only if*

a) $i_* : \pi_1(F) \to \pi_1(E)$ *is injective*

and

b) $p_* : \pi_1(E) \to \pi_1(B)$ *has a right inverse (which is a homomorphism).*

PROOF: Suppose that $\tilde{E}$ is a covering fibration. Then we have

$\pi_1(\tilde{E})$

$+B$ covers $F \to E \to B$

$(B) \to A \subset \pi_1(F)$

$\xrightarrow{\tilde{p}_*} \pi_1(B)$ is

epic.

$$\begin{array}{ccc} \tilde{F} & \xrightarrow{\pi} & F \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i} \\ \tilde{E} & \xrightarrow{\pi} & E \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow[1]{} & B \end{array}$$

where π denotes covering maps. From the exact ladder associated with this diagram, we have

$$\begin{array}{ccc} \pi_2(B) & \xrightarrow{\cong} & \pi_2(B) \\ \downarrow & & \downarrow{\scriptstyle d} \\ 0 & \longrightarrow & \pi_1(F). \end{array}$$

Thus d is trivial, hence $i_* : \pi_1(F) \to \pi_1(E)$ is injective. On the other hand $(p\pi)_* : \pi_1(\tilde{E}) \xrightarrow{\cong} \pi_1(B)$. So $(p\pi)^*$ has an inverse $j : \pi_1(B) \to \pi_1(\tilde{E})$. Then $\pi_* j : \pi_1(B) \to \pi_1(E)$ is the required inverse to $p_* : \pi_1(E) \to \pi_1(B)$.

Conversely, suppose a) and b) hold, Let $\tilde{E}$ be the covering space of E corresponding to a subgroup H of $\pi_1(E)$ such that $p_* : H \to \pi_1(B)$ is an isomorphism. Then $p' = p\pi : \tilde{E} \to B$ is a fibration with fibre F' and we

have the commutative diagram

$$\begin{array}{ccc} F' & \xrightarrow{\bar{\pi}} & F \\ {\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle i} \\ \tilde{E} & \xrightarrow{\pi} & E \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow[1]{} & B \end{array}$$

where $\bar{\pi}$ is the restriction of π.

Now we will show that $F' = \tilde{F}$. Notice that

$$p'_* : \pi_1(\tilde{E}) \to \pi_1(B)$$

is the composition of isomorphisms

$$\pi_1(\tilde{E}) \overset{\pi_*}{\cong} H \overset{p_*}{\cong} \pi_1(B)$$

and is therefore an isomorphism. Hence F' is path connected. Since $\bar{\pi} : F' \to F$ is a fibration with discrete fiber, it is a covering space [9, Theorem 10, p. 78].

We will show now that $\pi_1(F') = 0$. If $v \in \pi_1(F')$ then $i'_*(v) = 0$ by exactness. Hence $0 = \pi_* i'_*(v) = i_* \bar{\pi}_*(v)$. But $\bar{\pi}_*$ is injective since $\bar{\pi}$ is a covering and i_* is injective by hypothesis a). Therefore $v = 0$ as was to be shown.

Next we turn to question B in the case of universal coverings and oriented fibrations. We need a few technical remarks.

First, let us recall the definition of the first evaluation subgroup $G_1(F) \subset \pi_1(F)$. A homotopy $h_t : F \to F$ is called a *cyclic homotopy* if $h_0 = h_1 = 1_F$. The loop τ given by $\tau(t) = h_t(*)$ (where $* \in F$ is the base point) is called the *trace* of the homotopy. Then $G_1(F)$ consists of the subgroups of $\pi_1(F, *)$ whose elements are represented by loops which are traces of cyclic homotopies. For a list of properties of $G_1(F)$, see [5].

We also need the fact (see theorem 16.9, [3]) that for any fibre F, there exists a universal fibration $F \to E_\infty \to B_\infty$. Now the transgression $d_\infty : \pi_2(B_\infty) \to \pi_1(F)$ is related to $G_1(F)$ by the fact that $d_\infty(\pi_2(B_\infty)) = G_1(F)$. Now, as always, we assume that F is connected. Then let $\tilde{B}_\infty$ be the universal covering space of B_∞. Let $F \to D \to \tilde{B}_\infty$ denote the fibration classified by the covering map $\pi : \tilde{B}_\infty \to B_\infty$. It is easy to see that $d(\pi_2(\tilde{B}_\infty)) = G_1(F)$, where d is the transgression $d : \pi_2(\tilde{B}_\infty) \to \pi_1(F)$ It also is true that every oriented fibration may be regarded as a pullback of $F \to D \to \tilde{B}_\infty$. This follows since a classifying map $k : B \to B_\infty$ of an oriented fibration $F \to E \to B$ maps $\pi_1(B)$ to the identity of $\pi_1(B_\infty)$.

Hence, there is a lifting of k to $\tilde{k} : B \to \tilde{B}_\infty$ such that $k = \pi\tilde{k}$.

Now we can state and prove

THEOREM 2: *There exists a covering fibration of any oriented fibration with connected fibre F extending the universal covering space $\tilde{F}$ if and only if $G_1(F)$ is trivial.*

PROOF: Let $F \xrightarrow{i} D \xrightarrow{p} \tilde{B}_\infty$ be the fibration mentioned above. We shall show that there exists a covering fibration $\tilde{D} \to \tilde{B}_\infty$ which extends $\tilde{F}$ if and only if $G_1(F) = 0$. The pull back of the classifying map for $F \to E \to B$, denoted by $k : B \to B_\infty$, of $\tilde{F} \to D \to B$ will be the required covering fibration. That is, $\tilde{D} = k^*(D)$.

Now $\pi_1(\tilde{B}_\infty)$ is trivial. Thus (b) of theorem 1 is satisfied. On the other hand, $G_1(F) = 0$ implies that $d : \pi_2(\tilde{B}_\infty) \to \pi_1(F)$ is trivial. Hence $i_* : \pi_1(F) \to \pi_1(D)$ is injective. So condition (a) is satisfied and so we may apply theorem 1.

If we assume that $G_1(F) \neq 0$, then by theorem 1, we cannot find a covering fibration of $F \to D \to \tilde{B}_\infty$ extending $\tilde{F}$.

COROLLARY 3: *Let F be a compact polyhedron and $\chi(F) \neq 0$. Then there exists a covering fibration of any oriented fibration extending F, the universal covering F.*

PROOF: We know that $\chi(F) \neq 0$ implies that $G_1(F) = 0$.

COROLLARY 4: *Let F be a compact polyhedron with $\chi(F) \neq 0$. Then there exists a cross-section over the two-skeleton of the base space of any oriented fibration with fibre F.*

PROOF: By the corollary above, we have

$$\begin{tikzcd} \tilde{F} \arrow[r, "\pi"] \arrow[d] & F \arrow[d] \\ \tilde{E} \arrow[r, "\pi"] \arrow[d, "p"'] & E \arrow[d] \\ B \arrow[r] & B \end{tikzcd}$$

where B is a 2-dimensional CW complex.

The groups in which the obstructions to a cross-section of $\tilde{E} \xrightarrow{p} B$ lie must vanish. Thus there is a cross-section $c : B \to \tilde{E}$. Now πc is the required cross-section.

Now we shall study those fibrations with fibres RP^n, real projective space. We begin by giving alternative proofs to some results of Olum.

LEMMA 5 (Olum [8]) :

$$\xi_0(RP^n) \cong Z_2.$$

PROOF: Let $\hat{\xi}_0(S^n)$ denote the group of homotopy classes of base point preserving equivariant homotopy equivalences of S^n. We have an isomorphism

$$\xi_0(RP^n) = \hat{\xi}_0(S^n)$$

defined by sending f to the unique base point preserving map $\tilde{f}$ that covers f. By the method of proof of [2, TH. 2.5], the operation of suspension defines an isomorphism

$$\hat{\xi}_0(S^n) = \hat{\xi}_0(S^{n+1}), \quad n \geqq 1.$$

Therefore, we have

$$\xi_0(RP^n) = \xi_0(RP^{n+1}), \quad n \geqq 1.$$

Finally,

$$\xi_0(RP^1) = \xi_0(S^1) \cong Z_2.$$

COROLLARY 6.

$$\xi(RP^{2n+1}) \cong Z_2; \ \xi(RP^{2n}) \cong 0.$$

PROOF: Let L denote the space of self homotopy equivalences of RP^n. Let L_0 be the subspace of maps of L which preserve the base point. Now $G_1(RP^{2n}) = 0$ and

$$G_1(RP^{2n+1}) = \pi_1(RP^{2n+1});$$

see [4], corollary I.6 and theorem II.5. From the exact sequence arising from the fibration $L_0 \to L \xrightarrow{\omega} RP^n$, and noting that the image of ω_* is $G_1(RP^n)$, we see that $d : \pi_1(RP^n) \to \pi_0(L_0)$ is zero if n is odd and injective if n is even. Thus $i_* : \pi_0(L_0) \to \pi_0(L)$ is injective if n is odd and has kernel Z_2 if n is even. But $\pi_0(L_0) = \xi_0(RP^n) \cong Z_2$ by lemma 5. Also $i_* : \pi_0(L_0) \to \pi_0(L)$ must be onto since RP^n is connected. Thus $\xi(RP^n) = \pi_0(L)$ is Z_2 when n is odd and 0 when n is even.

COROLLARY 7. *Every fibration with fibre RP^{2n} is orientable.*

PROOF: Since $\xi(RP^{2n})$ is trivial, the fundamental group of the base must act trivially on $\xi(RP^{2n})$.

THEOREM 8. *Every fibration with fibre RP^{2n} is covered by an S^{2n} fibration with an involution.*

PROOF: The conclusion means that we can always find a covering fibration which extends the universal cover S^{2n} of RP^{2n}. But this follows immediately from theorem 2 and corollary 7. The extended total space must be a 2-fold covering of the original total space, and the deck transformation is the involution.

REMARK: Theorem 8 is not true for RP^{2n+1} even when the fibration is orientable. Consider the "universal oriented fibration",

$$RP^{2n+1} \xrightarrow{i} D \to \tilde{B}_\infty.$$

Now

$$i_* : \pi_1(RP^{2n+1}) \to \pi_1(D)$$

is trivial since

$$G_1(R^{2n+1}) = Z_2.$$

Thus in theorem 1 condition a) fails.

The next corollary is an application of theorem 8.

COROLLARY 9. *Let $RP^{2n} \to E \to B$ be a fibration. Then*

$$H^*(E; Z_2) = H^*(RP^{2n}; Z_2) \otimes H^*(B; Z_2)$$

as Z_2-vector spaces.

PROOF: Let $S^{2n} \to \tilde{E} \to B$ be an S^{2n}-fibration with an involution which covers $RP^{2n} \to E \to B$. Let $\lambda : \tilde{E} \to S_\infty$ be an equivariant map and let $\lambda : E \to RP^\infty$ denote the quotient map. Then the composition $RP^{2n} \xrightarrow{i} E \xrightarrow{\lambda} RP^\infty$ sends the generator $c \in H^1(RP^\infty; Z_2)$ to the generator $\hat{c} \in H^1(RP^{2n}; Z_2)$. A cohomology extension of the fiber

$$\theta : H^*(RP^{2n}; Z_2) \to H^*(E; Z_2)$$

is defined by $\theta(\hat{c}^t) = \lambda^*(c^t)$, $t \geqq 0$. The corollary now follows from the Leray-Hirsch theorem [9, p. 257].

3. Arbitrary coverings

We shall restrict our attention to coverings of fibre *bundles*. Our technique can probably be extended to Hurewicz fibrations, or Dold fibrations, but the needed propositions have not been written down yet. First we answer question B for fibre bundles. Then follow applications.

Let G be a group of homeomorphisms of F onto itself. Let G^* be the self homeomorphisms of a covering of F, denoted $\tilde{F}$, which are liftings of homeomorphisms in G. Let $\Phi : G^* \to G$ be the map which takes a map $f \in G^*$ to the induced map $\Phi(f) \in G$. Then Φ is continuous and also is a homomorphism.

THEOREM 10: *If there exists a cross-section $c : G \to G^*$ to Φ which is also a homomorphism, then there is a covering bundle of any G-bundle with fibre F which extends $\tilde{F}$.*

PROOF: Let $F \to E \to B$ be a G-bundle and let $G \to E^* \to B$ be the associated principal G-bundle. Then consider the G-bundle $\tilde{F} \to E \times_G \tilde{F} \to B$ where G acts on $\tilde{F}$ by means of the cross-section c. Then $\tilde{E} = E \times_G \tilde{F}$ is the required covering. The projection $\pi : \tilde{E} \to E$ is given by

$$\pi(\langle e, \tilde{x} \rangle) = \langle e, \pi(\tilde{x}) \rangle \in E^* \times_G F = E.$$

The next theorem gives conditions for Theorem 10 to hold. We shall let G_e^* be the identity component of G^* and G_e the identity component of G.

THEOREM 11: *A covering bundle extending $\tilde{F}$ always exists if* a) $G_1(F) \subset \pi_1(\tilde{F})$, *and* b) *there is a homomorphism*

$$\bar{c} : G/G_e \to G^*/G_e^*$$

which is a cross-section to the homomorphism

$$\Phi : G^*/G_e^* \to G/G_e$$

induced by Φ.

PROOF: Note that $\Phi : G^* \to G$ is a fibration with a discrete fibre. The fibre over 1_F consists of the liftings of 1_F. Assume that $\tilde{f} \in G_e^*$ is a lifting of 1_F. Then the path from $\tilde{f}$ to $1_{\tilde{F}}$ induces a cyclic homotopy $h_t : F \to F$. The trace of h_t must represent an element in $\pi_1(\tilde{F})$ by a). Thus the trace lifts to a closed path in $\tilde{F}$. Thus $\tilde{f}$ has a fixed point and hence $\tilde{f} = 1_{\tilde{F}}$.

This fact allows us to conclude that G_e and G_e^* are isomorphic. Thus we may construct the cross-section required by theorem 10 over G_e, and condition b) allows us to extend the cross-section over the other components.

REMARK: If G is connected and $G_1(F) = 0$, we may extend any covering $\tilde{F}$ to a covering G-bundle of any arbitrary G-bundle.

COROLLARY 12: *If F is a compact polyhedron and $\chi(F) \neq 0$, then $\tilde{F}$ can be extended to a covering bundle for any fibre bundle with connected structural group.*

Let M be a closed topological manifold which is unorientable and let $\tilde{M}$ denote its oriented double covering. Let $0(M)$ be the subgroup of $\pi_1(M)$ of elements represented by orientation preserving loops. Then $0(M)$ is the subgroup of $\pi_1(M)$ corresponding to the oriented double covering $\tilde{M}$.

From this point on, we shall consider fibre bundles with fibre M and study covering bundles which extend $\tilde{M}$. The end result will yield the main application of these techniques, Theorem 15.

THEOREM 13: *For any fibre bundle with fibre M, a closed unorientable topological manifold, the oriented double covering $\tilde{M}$ extends to a covering* *bundle. In addition, the structural group of this covering bundle preserves the orientation of* $\tilde{M}$.

PROOF: Every homeomorphism $h: M \to M$ has two liftings $\tilde{M} \to \tilde{M}$, one of which preserves and the other reverses the orientation of $\tilde{M}$. Consider the correspondence i which sends every $h \in G$ to its orientation preserving lifting in G^*. It is easy to see that $i: G \to G^*$ is continuous, a homomorphism of groups and a cross-section of $\Phi: G^* \to G$. Thus theorem 10 is satisfied. The group of the covering fibre bundle is $i(G)$, so the second statement of the theorem is clearly true.

If we compare theorem 13 with condition a) of theorem 11, we are lead to conjecture that $G_1(M) \subset O(M)$. This is in fact true, as the following theorem shows.

THEOREM 14:

$$G_1(M) \subset O(M).$$

PROOF: Let $\alpha \in G_1(M)$. Then there is a cyclic homotopy $h_t: M \to M$ whose trace represents α. Now h_t lifts to a homotopy $\tilde{h}_t: \tilde{M} \to \tilde{M}$ and $\tilde{h}_1$ is a lifting of the identity. Since $\tilde{h}_1$ is homotopic to $1_{\tilde{M}}$, $\tilde{h}_1$ preserves the orientation on $\tilde{M}$. There are only two liftings of the identity and one of them reverses orientation. So $\tilde{h}_1 = 1_{\tilde{M}}$. Thus the trace of $\tilde{h}_t$ is a loop which covers the trace of $\tilde{h}_t$. Hence α must be in $O(M)$.

THEOREM 15: *Let* $M \to E \xrightarrow{\pi} B$ *be a fibre bundle with fibre a closed topological n-manifold and with structural group G. If* $\chi(M) \neq O \pmod{p}$, *where p is a prime, then*

$$\pi^*: H^*(B; Z_p) \to H^*(E; Z_p)$$

is injective.

PROOF: First consider the case where $\pi_1(B)$ operates trivially on $H^n(M; Z_p)$. Then the theorem is true if M is orientable or if $p = 2$ [6, Theorem 12]. Suppose now that M is unorientable and $p \neq 2$. Let $\tilde{M}$ denote the orientable double covering of M. By theorem 13, we have

$$\begin{tikzcd}
\tilde{M} \arrow[d] & M \arrow[d] \\
\tilde{E} \arrow[r] \arrow[d, "\tilde{\pi}"'] & E \arrow[d, "\pi"] \\
B \arrow[r, "1"'] & B
\end{tikzcd}$$

and $\pi_1(B)$ operates trivially on $H^n(\tilde{M}; Z_p)$. Now $\chi(\tilde{M}) = 2\chi(M) \neq 0 \pmod{p}$. Since $\tilde{M}$ is orientable it follows that $\tilde{\pi}^*$ is injective. Then by commutativity π^* is injective.

Now consider the case where $\pi_1(B)$ operates non-trivially on $H^n(M; Z_p)$ p odd. Let $K \subset \pi_1(B)$ denote the normal subgroup of index 2 consisting of elements which operate trivially on $H^n(M; Z)$. Let $\tilde{B} \xrightarrow{q} B$ denote the 2-fold covering which corresponds to K. We have

$$\begin{tikzcd}
M \arrow[d] & M \arrow[d] \\
\tilde{E} \arrow[r, "\tilde{q}"] \arrow[d, "\tilde{\pi}"'] & E \arrow[d, "\pi"] \\
\tilde{B} \arrow[r, "q"'] & B
\end{tikzcd}$$

where $\tilde{E} \xrightarrow{\tilde{\pi}} \tilde{B}$ is the bundle induced by q. Since $\pi_1(B)$ operates trivially on $H^n(M; Z_p)$ we have from the preceding paragraph that $\tilde{\pi}^*$ is injective. To show that π^* is injective it is now sufficient to show that

$$q^*: H^*(B; Z_p) \to H^*(\tilde{B}; Z_p)$$

is injective. There is the transfer map [10, Chapter 5]

$$\tau: H^*(\tilde{B}; Z_p) \to H^*(B; Z_p),$$

and τq^*, being multiplication by 2, is an isomorphism. Therefore q^* is injective. This completes the proof of the theorem.

This theorem was first noted by A. Borel in [1] with the extra hypotheses that the bundle is differentiable and oriented in some sense and M is oriented. In [6], the second author removed the differentiability hypothesis and the above theorem removes the orientability hypotheses.

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