FIBRE BUNDLES AND THE EULER CHARACTERISTIC

DANIEL HENRY GOTTlieB

1. Introduction

For any fibre bundle \( F \to i \xrightarrow{i} E \xrightarrow{p} B \) there are three important maps: the projection \( p \), the fibre inclusion \( i \), and the evaluation \( \omega : OB \to F \). In this paper we demonstrate formulas for each of these maps involving the Euler-Poincaré number of the fibre.

Let \( M \) be a compact topological manifold with possibly empty boundary \( \bar{M} \), \( \chi(M) \) the Euler-Poincaré number of \( M \), \( G \) any space of homeomorphisms of \( M \) with a continuous action on \( M \), \( \omega : G \to M \) the evaluation map for some base point, \( M \xrightarrow{i} E \xrightarrow{p} B \) any (locally trivial) fibre bundle, and \( L \subset B \) a (possibly empty) subcomplex of the \( CW \) complex \( B \).

Theorem A. For connected \( M \) and any coefficients

\[
\chi(M) \omega^* = 0 : \tilde{H}^*(M) \to \tilde{H}^*(G).
\]

Theorem B. There exists a transfer homomorphism \( \tau : H^*(E, p^{-1}(L)) \to H^*(B, L) \) such that \( \tau \circ p^* = \chi(M) \) \( 1 \) for any coefficients.

Theorem C. There exists a transfer homomorphism \( \tau : H_*(E, p^{-1}(L)) \to H_*(B, L) \) such that \( p_* \circ \tau = \chi(M) \) \( 1 \) for any coefficients.

Special cases of Theorem A were discovered by the author in [3] and [4]. Note that \( B \) and \( C \) reduce to the classical transfer theorem for covering spaces when \( M \) is a finite set of points. Borel proved a version of Theorem B for \( M \) a closed connected differentiable manifold and \( M \xrightarrow{i} E \xrightarrow{p} B \) an “oriented” fibre bundle with structural group acting differentially on \( M \) and cohomology groups with fields of coefficients whose characteristics does not divide \( \chi(M) \), [2]. This result was improved by the author in [1] and [3].

All these theorems are consequences of the next. Let \( \bar{E} \) be the subspace of \( E \) consisting of those points of \( E \) which are in the boundaries of the fibres containing them. Then \( (M, \bar{M}) \xrightarrow{i} (E, \bar{E}) \xrightarrow{p} B \) is a fibre pair. If \( \bar{M} \) is empty, then \( \bar{E} \) is empty.

Theorem D. Let \( M^n \) be orientable and connected, and assume \( \pi_1(B) \) acts

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trivially on $H^n(M^k, M; \mathbb{Z}) \cong \mathbb{Z}$. Then there exists a \( \chi \in H^n(E, E'; Z) \) such that \( i^*(\chi) = \chi(M)\mu \) where \( \mu \) generates $H^1(M, M; \mathbb{Z})$.

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2. Integration along the fibre

Here we record some well-known facts concerning integration along the fibre.

Suppose \((F, F') \to (E, E') \to B\) is a fibre pair, and \(L\) is a subcomplex of \(B\). Then the Serre spectral sequence converges to $H^*(E, E' \cup p^{-1}(L); G)$ and $E_{1}^{p,q} \cong H^q(B, L; [H^*(F, F'; G)])$.

Suppose $\pi(B)$ operates trivially on $H^n(F, F'; Z) \cong Z$ and $H^n(F, F'; Z) \cong 0$ for $i > n$. Then integration along the fibre is defined as the composition $p_n : H^n(E, E' \cup p^{-1}(L)) \to E_{1}^{n,n} \Rightarrow E_{n,n}^{\text{int}} \cong H^n(B, L; H^n(M, M'; G)) \cong H^n(B, L; G)$.

Integration along the fibre satisfies three properties:

a) If $E \to E' \to B$ are two fibrations, then $(q \circ p)_* = q_\ast \circ p_\ast$.

b) If we have a fibre square

\[
\begin{array}{ccc}
(F, F') & \xrightarrow{f} & (\tilde{F}, \tilde{F}') \\
\downarrow & & \downarrow \\
(E, E' \cup p^{-1}(L)) & \xrightarrow{f^\ast} & (\tilde{E}, \tilde{E}' \cup \tilde{p}^{-1}(L)) \\
\downarrow & & \downarrow \\
(B, L) & \xrightarrow{f} & (\tilde{B}, \tilde{L})
\end{array}
\]

and $(F, F')$ and $(\tilde{F}, \tilde{F}')$ both have cohomological dimension $n$, then $H^n(E, E' \cup p^{-1}(L)) \xrightarrow{f^\ast} H^n(\tilde{E}, \tilde{E}' \cup \tilde{p}^{-1}(L)) \xrightarrow{\tilde{p}_\ast} H^n(B, L; G)$ commutes, where $\tilde{\psi}$ is induced by $f^\ast$ and a homomorphism on the coefficient group corresponding to the map induced by $f^\ast$.

c) If $u \in H^*(B, L; G)$ and $v \in H^*(E, E'; G')$ then $p_n(p^\ast(u) \cup v) = u \cup p_n^\ast(v) + p_n^\ast(v)$, where $G$ and $G'$ pair to $G''$ and $p_n^\ast : H^*(E, E' \cup p^{-1}(L)) \to H^*(B, L)$, and $p_n^\ast : H^*(E, E') \to H^*(B)$.

Dually, we may define $p_n$ as the composition $p_n : H^*(E, E' \cup p^{-1}(L)) \to E_{1}^{n,n} \Rightarrow E_{n,n}^{\text{int}} \cong H^n(B, L; H^n(M, M'; G)) \cong H^n(B, L; G)$.

3. Proof of Theorem D

Let $G$ be a group of orientation-preserving homeomorphisms on $M$ with compact-open topology acting transitively on $M = M \setminus \mathcal{M}$. Let $H$ be the subgroup of $G$ leaving the base point $* \in \mathcal{M}$ fixed. We take $\ast \in \mathcal{M}$.

Consider the universal principal bundle $G \to E_0 \to B_0$. Then the classifying space for $H$ is $B_H = E_0 \times_0 \mathcal{M}$ since $G/H = \mathcal{M}$. Let $\tilde{B}_H$ denote $E_0 \times_0 \mathcal{M}$, and let $\tilde{B}_0$ denote $E_0 \times_0 \mathcal{M}$. We have the following diagram of fibre squares:

\[
\begin{array}{ccc}
& & M \downarrow & \\
& & \downarrow & \\
& & \pi^*(\tilde{B}_H) \xrightarrow{j} \pi^*(\tilde{B}_H) \xrightarrow{\pi^*} B_H \xrightarrow{\pi} B_0 \\
\end{array}
\]

Here $j$ and $\tilde{j}$ are inclusion maps.

**Lemma 1.** Regarding $\tilde{j}$ as a map of pairs $\tilde{j} : (\pi^*(\tilde{B}_H), \pi^*(\tilde{B}_H)) \to (\pi^*(\tilde{B}_H), \pi^*(\tilde{B}_H))$.

Then $j$ and $\tilde{j}$ are homotopy equivalences.

**Lemma 2.** $(\pi^*(\tilde{B}_H), \pi^*(\tilde{B}_H)) = (E_0 \times_0 M, E_0 \times_0 \mathcal{M})$.

**Proof.**

\[
\begin{array}{ccc}
M & \xrightarrow{\pi} & M \\
\downarrow & & \downarrow \\
E_0 \times_0 M & \xrightarrow{j} & E_0 \times_0 \mathcal{M} = \tilde{B}_H \\
\downarrow & & \downarrow \\
E_0 \times_0 \mathcal{M} & = E_0 \times_0 \mathcal{M} & \xrightarrow{\pi} \tilde{B}_H = B_0 \\
\end{array}
\]
The existence of this fibre square implies that \( E_0 \times_H M = \pi^*(\tilde{B}_H) \).

Since \( M \) is oriented, \( Z \cong H^n(M, M - *) \xrightarrow{i^*} H^n(M, \tilde{M}) \) is an isomorphism where \( i \) is inclusion. Thus by Lemmas 1 and 2 and the naturality of integration along the fibre (§ 2(b)) we have the following commutative diagram:

\[
\begin{array}{c}
H^*(E_0 \times_H M, E_0 \times_H (M - *)) \xrightarrow{i^*} H^*(\pi^*(\tilde{B}_H), \pi^*(\tilde{B}_H)) \cong H^*(\pi^*(\tilde{B}_H), \pi^*(\tilde{B}_H)) \\
\cong H^*(\pi^*(\tilde{B}_H), \pi^*(\tilde{B}_H)) \xrightarrow{i^*} H^*(\tilde{B}_H, \tilde{B}_H) \\
\cong H^*(\tilde{B}_H, \tilde{B}_H) \xrightarrow{i^*} H^*(\tilde{B}_H, \tilde{B}_H) \\
(2) \end{array}
\]

Note that \( i^* \) is an isomorphism because the fibre of the fibre pair \( (E_0 \times_H M, E_0 \times_H (M - *)) \) is \( (M, M - *) \) which has the cohomology of \( (R^n, R^n - 0) \); thus the spectral sequence for \( i \) takes a very simple form, and \( i^* \) may be thought of as the Thom isomorphism.

Now we define \( U \in H^*(E_0 \times_H M, E_0 \times_H (M - *)) \) by the equation \( p_1(U) = 1 \). Define \( U_i \in H^*(\pi^*(\tilde{B}_H), \pi^*(\tilde{B}_H)) \) by \( U_i = (i^*)^{-1}i^*U \). Then \( \pi_1(U_i) = 1 \in H^*(\tilde{B}_H) \) by diagram (2).

We have the fibre square

\[
\begin{array}{c}
(M, \tilde{M}) \xrightarrow{i} (M, \tilde{M}) \\
\downarrow \downarrow \\
M \times (M, \tilde{M}) \xrightarrow{(i^*)^{-1}i^*} (\pi^*(\tilde{B}_H), \pi^*(\tilde{B}_H)) \\
\downarrow i^* \downarrow \tilde{B}_H \\
M \xrightarrow{i} \tilde{B}_H
\end{array}
\]

arising from the fibre inclusion \( M \xrightarrow{i} \tilde{B}_H \to B_0 \), and restricting diagram (2) to the bundles over the fibres yields

\[
\begin{array}{c}
H^*(\tilde{M} \times M, M \times M - *) \xrightarrow{1 \times i^*} H^*(\tilde{M} \times M, M \times M) \cong H^*(M \times M, M \times M) \\
\cong H^*(M \times M, M \times M) \xrightarrow{i^*} H^*(M) \\
(4) \end{array}
\]

where \( \Delta \) denotes the diagonal.

Define \( U \in H^*(\tilde{M} \times M, M \times M - *) \) by \( p_1(U) = 1 \), and define \( U_i \in H^*(M \times M, M \times M) \) as image of \( U \).

Now let \( T : X \times Y \to Y \times X \) stand for the twisting map. Noting that

\[
T^*(\pi^*(\tilde{B}_H)) \to \pi^{-1}((\tilde{B}_H, \tilde{B}_H)) \text{ arises from the restriction of the twisting map to } \pi^*(\tilde{B}_H) \subset \tilde{B}_H \times \tilde{B}_H, \text{ we have a commutative diagram:}
\]

\[
\begin{array}{c}
\begin{array}{c}
\xrightarrow{\pi^*(\tilde{B}_H)} \pi^*(\tilde{B}_H) \to \pi^*(\tilde{B}_H, \\
\downarrow i^* \downarrow \pi^*(\tilde{B}_H), \pi^*(\tilde{B}_H)) \cong \pi^*(\tilde{B}_H) \to \pi^*(\tilde{B}_H) \\
\end{array}
\end{array}
\]

(5)

where \( i \) comes from the fibre square (3).

Define \( U_2 \in H^*(\pi^*(\tilde{B}_H), \pi^*(\tilde{B}_H), \pi^*(\tilde{B}_H)) \) by \( U_2 = \pi^{-1}i^*U_1 \). Similarly define \( U_2 \in H^*(M \times M, M \times M, M) \). Then the naturality of integration along the fibre and diagram (5) implies that \( U_1, U_2 \), and \( U_3 \) defined in the universal case pull back under inclusion to \( U_1, U_2 \), and \( U_3 \) defined in the product case.

Now consider \( U_1 \cup U_2 \in H^*(M \times M, M \times M) \). We have a relative fibre bundle pair

\[
(M, \tilde{M}) \to (M \times M, M \times M) \xrightarrow{\pi} (M, M),
\]

and we may define integration along the fibre \( \pi_1 : H^*(M \times M, M \times M) \to H^{*-n}(M, M) \). In this situation, \( \pi_1 \) is the same as the slant product with the fundamental class \( z \in H_4(M, M) \) (that is, \( \pi_1(z) = z/2 \)). We call \( \chi = \pi_1(U) \cup U_1 \) the Euler class in \( H^*(M, M, M) \). This definition is easily seen to agree with that of Spanier [5, p. 347]. Thus we have \( \chi = \chi(M) \mu \in H^*(M, M) \) where \( \mu \) is the appropriately chosen generator.

On the other hand we have

\[
U_1 \cup U_2 \in H^*(\pi^*(\tilde{B}_H), \pi^*(\tilde{B}_H)) \to (\pi^*(\tilde{B}_H), \pi^*(\tilde{B}_H)).
\]

Note that \( T(\pi^*(\tilde{B}_H)) = \pi^{-1}(\tilde{B}_H) \). Thus we are lead to consider the relative fibre bundle pair

\[
(M, \tilde{M}) \to (\pi^*(\tilde{B}_H), \pi^*(\tilde{B}_H)) \to (\pi^*(\tilde{B}_H), \pi^*(\tilde{B}_H)).
\]

Thus we have integration along the fibre

\[
\pi_1 : H^*(\pi^*(\tilde{B}_H), \pi^*(\tilde{B}_H)) \to H^{*-n}(\tilde{B}_H, \tilde{B}_H).
\]

Define the Euler class \( \chi = \pi_1(U_1 \cup U_2) \in H^*(\tilde{B}_H, \tilde{B}_H) \). By naturality of \( \pi_1 \), we see that \( i^*(\chi) = \chi(M) \mu \) for \( i : (M, \tilde{M}) \to (\tilde{B}_H, \tilde{B}_H) \), the fibre inclusion.

Since \( (M, \tilde{M}) \to (\tilde{B}_H, \tilde{B}_H) \) is the universal bundle pair for bundle pairs \( (M, \tilde{M}) \to (E, E) \to B \) with structural group preserving the orientation of \( (M, \tilde{M}) \), we always can find a fibre square.
(6) \[(M, \tilde{M}) \xrightarrow{i} (M, M) \]
\[\xrightarrow{\tilde{f}} \quad \quad \quad (E, \tilde{E}) \quad \xrightarrow{\tilde{f}} (\tilde{B}_M, \tilde{B}_M) \]
\[\xrightarrow{\pi} \quad \quad \quad B \quad \xrightarrow{f} \quad \tilde{B}_M \]

Define \(\chi \in H^*(E, \tilde{E})\) by \(\chi = \tilde{f}^*(\chi')\). It is clear that \(i^*(\chi) = \chi(M)\mu\), so Theorem D is proved.

Note that every possible \(\tilde{f}\) which arises in diagram (6) must be fibrewise homotopic to any other [4], so \(\chi\) is uniquely defined.

4. Proof of Theorem A

It is clear that Theorem A would be true in general if we can prove Theorem A for the case where \(G\) is the identity component of the group of homeomorphisms of \(M\). So we make that assumption.

First we shall prove Theorem A when \(M\) is an oriented manifold. We have the fibre square

\[
\begin{array}{ccc}
G \times (M, \tilde{M}) & \xrightarrow{\tilde{\omega}} & (M, \tilde{M}) \\
\downarrow i \times 1 & & \downarrow \\
E_0 \times (M, \tilde{M}) & \xrightarrow{\phi} & (E_0 \times_0 M, E_0 \times_0 \tilde{M}) \\
\downarrow & & \downarrow \\
B_0 & \xrightarrow{\phi} & B_0
\end{array}
\]

where \(\phi\) is the action of \(G\) on \(M\), and \(\phi\) takes \((e, x) \mapsto \langle e, x \rangle\). Since \(G\) is connected, we may apply Theorem D to the fibration on the right. Thus \(\omega^*(\chi(M)\mu) = (i \times 1)^* \phi^*(\chi')\). Since \(E_0\) is contractible, we see that

\[
\omega^*(\chi(M)\mu) = 1 \times (\chi(M)\mu) \in H^*(G \times (M, \tilde{M}); Z).
\]

Let \(\alpha \in H^i(M; G)\) be any element for \(i > 0\). Then \(\alpha \cup (\chi(M)\mu) \in H^{*+i}(M, \tilde{M}; G) = 0\). Thus

\[
0 = \omega^*(\alpha \cup (\chi(M)\mu) = \omega^*(\alpha) \cup (\omega^*(\chi(M)\mu))
\]

\[
= ((\omega^*(\alpha) \times 1) + \text{other terms}) \cup (1 \times (\chi(M)\mu))
\]

\[
= \omega^*(\alpha) \times (\chi(M)\mu) + \text{other terms} \cup (1 \times \chi(M)\mu)
\]

\[
= \omega^*(\alpha) \times (\chi(M)\mu) = \chi(M)\omega^*(\alpha) \times \mu
\]

Hence \(\chi(M)\omega^*(\alpha) = 0\) when \(M\) is oriented.

Now we assume that \(M\) is unoriented. Let \(\tilde{M}\) be the oriented double covering of \(M\), and \(D\) the mapping cylinder of the projection \(\tilde{M} \to M\). Then \(D\) is a manifold with boundary. We may think of \(G\) as acting on \(\tilde{M}\) by lifting every homeomorphism \(h : M \to M\) to that lifting \(\tilde{h} : \tilde{M} \to \tilde{M}\) which preserves orientation. Then \(G\) acts on \(D\) as a group of homeomorphisms by \(g(x, t) = (\tilde{g}(t), t)\).

Thus we obtain the following commutative diagram:

\[
\begin{array}{ccc}
G & \xrightarrow{\omega} & D \\
\downarrow & & \downarrow \\
M & \xrightarrow{f} & D
\end{array}
\]

Since the inclusion \(i\) is a homotopy equivalence, Theorem A holds for \(G \to M\) if it holds for \(G \to D\). But this is the case as follows from the following lemma.

**Lemma 3.** \(D\) is orientable, and \(G\) preserves the orientation.

**Proof.** First assume that \(M\) is closed. Then \(\tilde{D} = \tilde{M}\) and is orientable. An examination of the homology exact sequence of the pair \((D, \tilde{D})\) shows that \(H_{*+1}(D, \tilde{D}) = Z\). So \(D\) is orientable.

Now assume that \(\tilde{M}\) has nonempty boundary \(\tilde{M}\). Then \(\tilde{D} = \tilde{M} \cup D(\tilde{M})\) where \(D(\tilde{M})\) is the mapping cylinder of \(\tilde{M} \to M\). Now either \(D(\tilde{M})\) is \(M \times I\) in case \(\tilde{M}\) is orientable or it is the mapping cylinder of the bundle covering of \(M\). In either case \(D(\tilde{M})\) is orientable. Thus \(D\) is orientable. Then the homology exact sequence of \((D, \tilde{D})\) implies that \(D\) is orientable. It is easily seen that \(G\) preserves the orientation.

5. Proof of Theorem B

We first prove Theorem B for the case when \(M\) is connected and orientable and \(\pi\) acts trivially on \(H^*(M^*, \tilde{M}) \cong Z\) in the fibration \((M, \tilde{M}) \to (E, \tilde{E}) \to B\).

Define \(\tau : H^*(E, \tilde{E}; G) \to H^*(B; G)\) by letting \(\tau(\alpha) = \pi(\alpha \cup \chi)\).

**Lemma 4.** \(\tau \circ p^*(\alpha) = \chi(M)\alpha\) for all \(\alpha \in H^*(B, L; G)\).

**Proof.** From the fibre square

\[
\begin{array}{ccc}
(M, \tilde{M}) & \xrightarrow{i} & (M, \tilde{M}) \\
\downarrow & & \downarrow \\
(M, \tilde{M}) & \xrightarrow{p} & (E, \tilde{E}) \\
\downarrow & \uparrow & \downarrow \\
\tilde{M} & \xrightarrow{\chi} & B
\end{array}
\]
we have \( \pi_1(\chi) = \pi_1^G(\chi) \) by identifying \( H^*(*) \) with \( H^*(B) \). So \( \pi_1(\chi) = \pi_1^G(\chi) \). Hence \( \tau \circ p^*(\alpha) = \pi_1^G(\chi) \). Hence \( \tau \circ p^*(\alpha) \cap \chi = \chi(\alpha) \). Hence \( \tau \circ p^*(\alpha) \cap \chi = \alpha \cap \gamma(\chi) \). Here \( \gamma(\chi) \) is the Euler class of \( B \).

From now on we shall suppress \( L \) and \( p^*(L) \) in our notation.

Next we shall show Theorem B is true for \( M \) unoriented and connected. Let \( D \) be the mapping cylinder as in diagram (8). The projection \( q : D \to M \) is equivariant with respect to the action of \( G \). Thus we get a fibre square

\[
\begin{array}{cccc}
D & \xrightarrow{q} & M \\
\downarrow & & \downarrow \\
E & \xrightarrow{\bar{q}} & E \\
\downarrow & & \downarrow \\
B & \xrightarrow{p} & B.
\end{array}
\]

(10)

The left fibration satisfies the previous case since \( D \) is oriented and \( G \) preserves the orientation by Lemma 3, so there exists a transfer \( \tau : H^*(E ; G) \to H^*(B ; G) \). Define \( \tau : H^*(E ; G) \to H^*(B ; G) \) by \( \tau = \tau \circ \bar{q}^* \). Then \( \tau \circ p^* = \tau \circ \bar{q}^* \circ \pi_1^G = \pi_1^G(\gamma(D)) = \chi(M) \).

Now we assume that \( M \) is orientable and connected but that \( \pi_1\) does not act trivially on \( H^*(M ; \mathbb{Z}) \). Then we obtain the commutative diagram

\[
\begin{array}{cccc}
M \times P^n & \xrightarrow{\tau} & M \\
\downarrow & & \downarrow \\
E \times P^n & \xrightarrow{\tau} & E \\
\downarrow & & \downarrow \\
B & \xrightarrow{p} & B
\end{array}
\]

where \( P^n \) is the real projective plane, and \( \pi \) is projection on the first factor. The fibre bundle on the left satisfies the above case since \( M \times P^n \) is orientable. Thus there exists a transfer \( \tau : H^*(E ; P^n) \to H^*(B ; G) \). Define \( \tau : H^*(E ; G) \to H^*(B ; G) \) by \( \tau = \tau \circ \pi^G \). Then \( \tau \circ p^* = \tau \circ \pi^G \circ p^n = \tau \circ p^n = \chi(M) \).

Now assume that \( M \) is not connected. Then the fibre bundle \( M \to E \to B \) factors through the fibre bundles \( E \to \tilde{B} \to B \), where \( \tilde{B} \) is an \( N \)-fold covering of \( B \), and \( M \) is \( N \) disjoint copies of \( M_0 \). Thus we have a transfer for \( M_0 \to E \to B \); call it \( \tau_0 \). Also we have the classical transfer for the covering \( \tau_1 \). Define \( \tau : H^*(E ; G) \to H^*(B ; G) \) by \( \tau = \tau_1 \circ \tau_0 \). Then \( \tau \circ p^* = \tau_1 \circ \tau_0 \circ p^n \circ p^G \).

In the case where \( E \) is not connected, we obtain a transfer for each component of \( E \). Then we sum them to obtain the transfer for \( E \to B \). Finally, if \( B \) is not connected, (we assume that each fibre of \( E \to B \) is \( M_0 \)), then the direct sum of the transfers over each component of \( B \) will yield the transfer we seek.

6. Proof of Theorem C and remarks

We begin as before, by assuming that \( E \) and \( M \) are connected and \( M \) is orientable, and that \( \pi_1\) preserves orientation. Then we have the Euler class \( \chi \in H^*(E , \mathbb{Z}) \). Define the transfer \( \epsilon : H_*(B ; L) \to H_*(E , \mathbb{Z}) \) by \( \epsilon = \chi \). Then \( \epsilon \circ \pi^G = \chi \circ \pi^G \). Then \( \epsilon \circ \pi^G = \chi \circ \pi^G \).

Now we assume that \( M \) is orientable and connected but that \( \pi_1\) does not act trivially on \( H^*(M ; \mathbb{Z}) \). Then we obtain the commutative diagram

\[
\begin{array}{cccc}
M \times P^n & \xrightarrow{\tau} & M \\
\downarrow & & \downarrow \\
E \times P^n & \xrightarrow{\tau} & E \\
\downarrow & & \downarrow \\
B & \xrightarrow{p} & B
\end{array}
\]

where \( P^n \) is the real projective plane, and \( \pi \) is projection on the first factor. The fibre bundle on the left satisfies the above case since \( M \times P^n \) is orientable. Thus there exists a transfer \( \tau : H^*(E ; P^n) \to H^*(B ; G) \). Define \( \tau : H^*(E ; G) \to H^*(B ; G) \) by \( \tau = \tau \circ \pi^G \). Then \( \tau \circ p^* = \tau \circ \pi^G \circ p^n = \tau \circ p^n = \chi(M) \).

Now assume that \( M \) is not connected. Then the fibre bundle \( M \to E \to B \) factors through the fibre bundles \( E \to \tilde{B} \to B \), where \( \tilde{B} \) is an \( N \)-fold covering of \( B \), and \( M \) is \( N \) disjoint copies of \( M_0 \). Thus we have a transfer for \( M_0 \to E \to B \); call it \( \tau_0 \). Also we have the classical transfer for the covering \( \tau_1 \). Define \( \tau : H^*(E ; G) \to H^*(B ; G) \) by \( \tau = \tau_1 \circ \tau_0 \). Then \( \tau \circ p^* = \tau_1 \circ \tau_0 \circ p^n \circ p^G \).

In the case where \( E \) is not connected, we obtain a transfer for each component of \( E \). Then we sum them to obtain the transfer for \( E \to B \). Finally, if \( B \) is not connected, (we assume that each fibre of \( E \to B \) is \( M_0 \)), then the direct sum of the transfers over each component of \( B \) will yield the transfer we seek.

6. Proof of Theorem C and remarks

We begin as before, by assuming that \( E \) and \( M \) are connected and \( M \) is orientable, and that \( \pi_1\) preserves orientation. Then we have the Euler class \( \chi \in H^*(E , \mathbb{Z}) \). Define the transfer \( \epsilon : H_*(B ; L) \to H_*(E , \mathbb{Z}) \) by \( \epsilon = \chi \). Then \( \epsilon \circ \pi^G = \chi \circ \pi^G \). Then \( \epsilon \circ \pi^G = \chi \circ \pi^G \).
References


Purdue University