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FIBERING SUSPENSIONS

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1. **Introduction.** Let the term *compact fibration* denote a Hurewicz fibration $F \xrightarrow{i} E \xrightarrow{p} B$ such that B and F are homotopy equivalent to finite connected CW complexes. We are concerned with the following question: Under what circumstances is a suspension, ΣX, the total space of a "non trivial" compact fibration? (A *trivial* fibration is one in which either F or B is contractible.)

This is a natural question to ask, since the universal fibration for a connected Lie group is a direct limit of compact fibrations whose total spaces are suspensions. The same is true for finite H-spaces. Thus compact fibrations of suspensions play an important role in topology and especially in the study of finite H-spaces.

Work on compact fibrations where ΣX is a sphere was begun by A. Borel [4] and Eckmann, Samelson, and G. W. Whitehead [7] as well as others. Finally, W. Browder in [5] virtually solved the problem by showing that the fibre F is the homotopy type of S^1 or S^3 or S^7. Browder's argument depended upon the fact that the fibre inclusion $F \xrightarrow{i} S^n$ is inessential. This then implies that F is an H-space and then Browder used the Bockstein Spectral Sequence to show that F is a sphere.

Browder's theorem, in conjunction with the universal fibration examples, hints that for a compact fibration $F \to \Sigma X \to B$ the fibre F should always be a finite H-space. However in §2c we produce a compact fibration of a suspension whose fibre is a Moore space and not an H-space. On the other hand we show that a mild condition on the base $(\widetilde{H}_*(\Omega B;Q) \neq 0)$ implies that F must have many properties in common with an H-space.

The main tool is a generalization of the Bott-Samelson theorem which states that the Pontrajagin ring $H_*(\Omega B;Q)$ contains a free tensor algebra which is isomorphic to $H_*(\Omega\Sigma X;Q)$. (See Theorem 3.) This quickly leads to results such as $\chi(F) = 0$.

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In §2, we give three different methods for constructing compact fibrations of suspensions. The first two are closely related and are generalizations of the original Hopf fibration $S^1 \to S^3 \to S^2$. The third method involves Moore spaces and gives rise to the example mentioned above with the essential fibre inclusion. These families of examples are the only ones known at this time.

In §3 we study the main tool of this paper, the generalized Wang exact sequence. This leads to the Main Theorem which states that for almost all compact fibrations $F \to \Sigma X \xrightarrow{p} B$, the homomorphism $(\Omega p)_*: H_*(\Omega\Sigma X;K) \to H_*(\Omega B;K)$ is injective for K a field of coefficients. A corollary of the proof is the Bott-Samelson theorem.

In §4 we use the main theorem to find numberous conditions implied by the existence of a compact fibration of a suspension. For example, in almost all cases, $H_*(\Omega B) \cong H_*(\Omega\Sigma X) \otimes H_*(F)$ as groups with fields of coefficients.

We may ask the following question: Which compact CW complexes are "prime"? That is, for which X is there no nontrivial compact fibration $F \to X \to B$ which "factors" the total space. Browder, [4], shows that even dimensional spheres are primes. In §4 we show that any compact simply connected ΣX is prime if $\chi(\Sigma X) \neq 0$ and $H_*(\Sigma X;Z)$ has no torsion.

In §4 we study compact fibrations of the form $\Sigma Y \to \Sigma X \to B$ and show that ΣY must be an odd dimensional sphere. Also compact fibrations of the form $F \to \Sigma X \to S^n$ exist only when $\Sigma X = S^3$, S^7 or S^{15}.

In §5 we consider the case of a compact fibration of a suspension $F \xrightarrow{i} \Sigma X \xrightarrow{\omega} B$ where the map ω: $\Sigma X \to B$ is an *evaluation* map. By an *evaluation map* we mean a map which factors through the identity component of B^B and the map from $B^B \to B$ given by evaluating at a base point. The Hopf fibration $S^1 \to S^3 \to S^2$ is an example of a compact fibration of a suspension whose projection is an evaluation map. We will show that it is the only non-trivial example. This has the following immediate corollary which expresses the oppositness of suspensions and H-spaces: Any map from a suspension to a finite H-space cannot have a compact connected homotopy theoretic fibre, unless the fibre is contractible.

§2. **Examples.** There are three known classes of examples. The first two are closely related and depend upon the fact that the join, $X * Y$, is homotopy equivalent to $\Sigma(X \wedge Y)$.

A. Suppose that G is an H-space with multiplication μ: $G \times G \to G$. The Hopf construction m: $G * G \to \Sigma G$ given by $m(\langle g,t,h\rangle) = \langle t,\mu(g,h)\rangle$ gives rise to a fibration

$$G \xrightarrow{i} G * G \xrightarrow{m} \Sigma G$$

called the Hopf fibration, or the Sugawara fibration, for (G,μ). (see [18], or Stasheff [17], page 3, or Rutter [15].) For G homotopy equivalent to a connected finite CW complex, a compact fibration results.

B. Suppose that $G \to E \to B$ and $G \to E' \to B'$ are principal G-bundles. Then G acts diagonally on $E * E'$ and we obtain a principal G bundle $G \to E * E' \to (E * E')/G$. If both the original bundles were compact fibrations the join construction usually gives rise to a compact fibration. Note that if G is homotopy equivalent to a compact connected CW complex, the universal bundle $G \to E_G \to B_G$ is the direct limit of compact fibrations of suspensions.

The third family of examples are made out of Moore spaces. A Moore space M(G,n) is a connected complex such that $\tilde{H}_i(M(G,n);Z) = 0$ for $i \neq n$ and $\tilde{H}_n(M(G,n);Z) \cong G$. We will need the fact that $\Sigma(M(G,n)) = M(G,n+1)$.

C. Now suppose that G and G′ are both finite abelian groups and the order of G is relatively prime to the order of G′. Let $n > 1$. Then M(G,n) and M(G′,n) are compact CW complexes and $M(G \oplus G',n)$ is homotopy equivalent to $M(G,n) \times M(G',n)$, which can readily be seen from the Künneth formula. Thus

$$M(G,n) \to M(G,n) \times M(G',n) \xrightarrow{\text{Projection}} M(G',n)$$

is a compact fibration of a suspension.

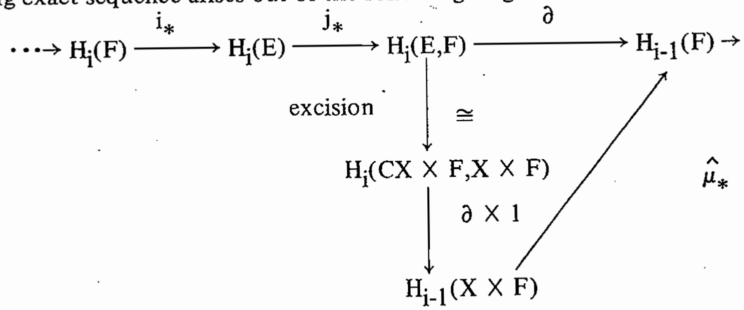
Note that examples A and B have the property that the fibre inclusions are inessential, and that the fibres are H-spaces. Example C has an essential fibre inclusion and a fibre which is not an H-space, in fact the Euler-Poincare number of the fibre equals 1, and the rational homology groups of the base and fibre are trivial.

§3. **The Wang exact sequence and the main theorem.** The main tool of this paper is the generalized Wang exact sequence for a fibration $F \xrightarrow{i} E \xrightarrow{p} \Sigma X$ over a suspension. A good account of this may be found in Spanier, [16, page 455]. We shall recall it here.

Any fibration over a suspension is classified by its clutching map

$$\hat{\mu}: X \times F \to F.$$

The Wang exact sequence arises out of the following diagram



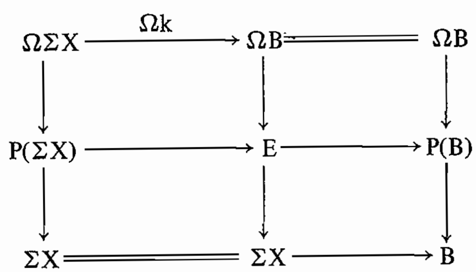
We begin by observing a fact about clutching maps of principal fibrations. From this and the Wang exact sequence we obtain the Main Theorem, and the Bott-Samelson theorem.

We let $m: \Omega B \times \Omega B \to \Omega B$ be loop multiplication and $\rho: X \to \Omega\Sigma X$ be the adjoint of the identity map.

THEOREM 1. *Given a principal fibration $\Omega B \to E \to \Sigma X$ with classifying map $k: \Sigma X \to B$, the clutching map $\hat{\mu}: X \times \Omega B \to \Omega B$ is given by*

$$\hat{\mu}: X \times \Omega B \xrightarrow{\rho\times 1} (\Omega\Sigma X) \times (\Omega B) \xrightarrow{(\Omega k)\times 1} (\Omega B) \times (\Omega B) \xrightarrow{m} \Omega B.$$

PROOF. Consider the commutative diagram.



Now ρ represents $\partial[1]$, the image of $1_{\Sigma X}$ under the homomorphism

$$\partial: [\Sigma X,\Sigma X] \to [X,\Omega\Sigma X].$$

Hence $\partial[k] = [(\Omega k)\circ\rho]$. But $\partial[k]$ is the homotopy class of the adjoint of the clutching map $\hat{\mu}$ (see [12], Proposition 1.3, for example).

THEOREM 2. (Bott-Samelson) *Let X be connected. The Pontrajagin ring of $H_*(\Omega\Sigma X;K)$ is isomorphic to a free tensor algebra with a unit, generated by a basis for*

$\tilde{H}_*(X;K)$.

REMARK. The original proof used the Serre spectral sequence. Other proofs have been published by Rutter [14] and Berstein [3].

Our proof uses the generalized Wang exact sequence. Although this proof is published for the first time here, it was essentially known to M. Barrett and S. Gitler.

PROOF. Consider the path fibration $\Omega\Sigma X \to P \to \Sigma X$. From the Wang exact sequence and the fact that $H_*(P;K)$ is trivial, we easily see that

$$\hat{\mu}_*: \tilde{H}_*(X;K) \otimes H_*(\Omega\Sigma X;K) \to H_*(\Omega\Sigma X;K)$$

is an isomorphism. Since the clutching map $\hat{\mu} = m(\rho\times 1)$ by Theorem 1, we see that ρ_* is injective and hence $\tilde{H}_*(X;K)$ may be identified with its image under ρ_* in $H_*(\Omega X;K)$. Hence we may regard $\hat{\mu}_*$ as m_* restricted to $\tilde{H}_*(X;K) \otimes H_*(\Omega\Sigma X;K)$, which is contained in $H_*(\Omega\Sigma X;K) \otimes H_*(\Omega\Sigma X;K)$. Now the fact that $\hat{\mu}_*: \tilde{H}_*(X;K) \otimes H_*(\Omega\Sigma X;K) \to H_*(\Omega\Sigma X;K)$ is an isomorphism is equivalent to the fact that $H_*(\Omega\Sigma X;K)$ is a free tensor algebra with unit on the generators of $\tilde{H}(X;K)$ under the ring multiplication given by m_*. Note that in proving this isomorphism it is necessary to assume that X is connected, so that $\tilde{H}_0(X;K) = 0$.

THEOREM 3. (The Main Theorem). *Let X and F be path connected. Suppose that*

$$F \xrightarrow{i} \Sigma X \xrightarrow{p} B$$

is a fibration and suppose that $H_i(F;K) \neq 0$ for only a finite number of i's. Suppose that $H_i(\Omega B;K) \neq 0$ for infinitely many i's, then $(\Omega p)_*: H_*(\Omega\Sigma X;K) \to H_*(\Omega B;K)$ is injective.

PROOF. From the usual sequence we obtain the principal fibration

$$\Omega B \xrightarrow{d} F \xrightarrow{i} \Sigma X$$

with clutching map

$$\hat{\mu}: X \times \Omega B \xrightarrow{\rho\times 1} \Omega\Sigma X \times \Omega B \xrightarrow{(\Omega p)\times 1} \Omega B \times \Omega B \xrightarrow{m} \Omega B$$

as in Theorem 1.

Consider the Wang exact sequence as given above. This is, in our notation here,

$$\begin{array}{ccccc} \to H_*(\Omega B) & \xrightarrow{d_*} & H_*(F) & \xrightarrow{j_*} & H_*(CX,X) \otimes H_*(\Omega B) \\ & & & & \big\downarrow \hat{\mu}_* o(\partial \times 1) \\ & & & & H_{*-1}(\Omega B) \longrightarrow \cdots \end{array}$$

where the coefficients of H_* are understood to be a field K. If i is large enough then $H_i(F) = H_{i-1}(F) = 0$ and

$$\hat{\mu}_* o(\partial \times 1)\colon \Sigma_{j+k=i} H_j(CX,X) \otimes H_k(\Omega B) \xrightarrow{\cong} H_{i-1}(\Omega B).$$

Thus for some nonzero $b \in H_i(\Omega B)$ for large enough i, we see that $\hat{\mu}_*(x \otimes b) \neq 0$ for all nonzero $x \in \tilde{H}_*(X)$. Hence $m_*(((\Omega p)_* o(\rho_*)(x)) \otimes b) \neq 0$. Hence $(\Omega p)_* o \rho_*$ is injective. Now by the *Bott-Samelson theorem* $H_*(\Omega\Sigma X)$ is a free algebra on the generators of $\tilde{H}_*(X)$, so an induction argument, based on the length of the product $x_1 x_2 \cdots x_n \in H_*(\Omega\Sigma X)$ and on the associativity of the Pontrajagin ring $H_*(\Omega B)$, establishes that $(\Omega p)_*$ is injective.

Let us consider the case where X is not connected. Then $\pi_1(\Sigma X) \cong$ Free group of rank c - 1 where c is the number of path-components of X. The conclusion of the Bott-Samelson theorem does not hold in this case, thus blocking the previous argument from applying here, however, we do get the following analogue of the main theorem.

THEOREM 4. *Let F and B be connected finite dimensional CW complexes. Suppose that $H_i(\Omega B;K) \neq 0$ for infinitely many i's. Let $F \to \Sigma X \xrightarrow{p} B$ be a fibration. Then $p_*\colon \pi_1(\Sigma X) \to \pi_1(B)$ is an isomorphism.*

PROOF. p_* is onto by the homotopy exact sequence and the fact that F is connected.

Now let $\tilde{B}$ be the universal covering of B. Then $\tilde{B}$ is not contractible since $\tilde{H}_i(\Omega\tilde{B};K) \not\cong 0$ for some $i > 0$. Hence $H_i(\Omega\tilde{B};K) \neq 0$ for infinitely many $i > 0$, since $\tilde{B}$ is finite dimensional. Consider the pullback

$$\begin{array}{ccc} F & \xrightarrow{1} & F \\ \downarrow & & \downarrow \\ E & \longrightarrow & \Sigma X \\ \downarrow & & \downarrow \\ \tilde{B} & \xrightarrow{p} & B \end{array}$$

Now, (1) E is a covering of ΣX corresponding to the subgroup of $\pi_1(\Sigma X)$ which is the kernel of p_*. Also (2), any covering of a space which is homotopy equivalent to a suspension, is itself homotopy equivalent to a suspension. So $E \simeq \Sigma Y$. Now a path component Y_α of Y gives rise to a generator $y_\alpha \in \tilde{H}_0(Y;K)$. Since $p_*\colon \pi_1(\Sigma Y) \to \pi_1(\tilde{B}) = 0$ is trivial, $y_\alpha \otimes 1$ maps to $\bar{y}_\alpha = 0$ under $\hat{\mu}_*\colon \tilde{H}_*(Y;K) \otimes H_*(\Omega\tilde{B};K) \to H_*(\Omega\tilde{B};K)$.

Now let $0 \neq b \in H_N(\Omega\tilde{B};K)$ where N is greater than the maximum of the dimensions of $\tilde{B}$ and F. Then $\hat{\mu}_*(y_\alpha \otimes b) = \bar{y}_\alpha \cdot b = 0$. But $\hat{\mu}_*$ is an isomorphism in dimension N, so $y_\alpha \otimes b = 0$ which implies that $y_\alpha = 0$. Thus $\tilde{H}_0(Y;K) = 0$ and Y is connected so $\pi_1(\Sigma Y) = 0$ so $\ker p_* = 0$.

The following example shows this theorem is false if the condition on $H_*(\Omega B;K)$ is not met. Let X be three points. Let F_2 be the free group on two generators. Then $\Sigma X = K(F_2,1)$. Let $p\colon \Sigma X \to S^1 \times S^1 = B$ be induced by the projection homomorphism $F_2 \to F_2/[F_2,F_2] \cong Z \oplus Z$. The fibre of p will be a $K(\pi,1)$ where $\pi \cong [F_2,F_2]$ is a free group. Hence the fibre has the homotopy type of a 1-dimensional connected complex. But $p_*\colon \pi_1(\Sigma X) \to \pi_1(B)$ is clearly not injective.

§4. **Applications of the main theorem.** In this section we shall always assume that $F \to \Sigma X \xrightarrow{p} B$ is a compact fibration, that is F and B will always be compact connected CW complexes.

THEOREM 5. *Suppose that $F \xrightarrow{i} \Sigma X \xrightarrow{p} B$ is a compact fibration with the assumptions that X is connected and $H_i(\Omega B;K) \neq 0$ for infinitely many i's. Then*

(i) *$H_*(\Omega B;K) \cong H_*(F;K) \otimes H_*(\Omega\Sigma X;K)$.*

(ii) *$d_*\colon H_*(\Omega B;K) \to H_*(F;K)$ is onto where $d\colon \Omega B \to F$ is the connecting map.*

(iii) *$i_*\colon H_*(F;K) \to H_*(\Sigma X;K)$ is trivial.*

PROOF. Consider the sequence of fibrations,

$$\cdots \to \Omega\Sigma X \xrightarrow{(\Omega p)} \Omega B \xrightarrow{d} F \xrightarrow{i} \Sigma X \xrightarrow{p} B.$$

Since $(\Omega p)_*$ is injective, we know that $(\Omega p)^*\colon H^*(\Omega B;K) \to H^*(\Omega\Sigma X;K)$ is onto. Hence the spectral sequence for the fibration

$$\Omega\Sigma X \xrightarrow{(\Omega p)} \Omega B \xrightarrow{d} F$$

collapses. As a consequence (i) and (ii) are true. Then (iii) follows since d_* is onto and

$i_* d_* = 0$.

COROLLARY 6. *Let $F \to \Sigma X \xrightarrow{p} B$ be a compact fibration such that X is connected and $\tilde{H}_*(B;Q) \neq 0$. Then either F is contractible or $\chi(F) = 0$.*

PROOF. Assume that $\chi(F) \neq 0$. Since X is connected, $\pi_1(\Sigma X) = 0$. Hence $d: \pi_2(B) \to \pi_1(F)$ is onto. But since $\chi(F) \neq 0$, the image of d must be zero (this follows from Theorem IV.1 of [8] and the fact that d factors through the induced homomorphism of an evaluation map). Thus $\pi_1(F) = 0$.

Next we shall show that $\tilde{H}_*(F;Z) \cong 0$. Since $\tilde{H}_*(B;Q) \neq 0$ and B is compact, we see that $\tilde{H}_*(B;K) \neq 0$ for every field K, and then by applying the Serre spectral sequence to the path fibration $\Omega B \to P \to B$, we see that $H_i(\Omega B;K) \neq 0$ for infinitely many i's. Thus $d_*: H_*(\Omega B;K) \to H_*(F;K)$ is onto by Theorem 5. But by a result of [2], reproduced here as Theorem 12, we see that $\chi(F) d_* = 0$. Hence for K = Q, this implies that $\tilde{H}_*(F;Q) = 0$. Hence $\chi(F) = 1$. Then $d_* = 0$ for every field K, hence $\tilde{H}_*(F;K) = 0$ for every field K. Since F is a finite CW complex, this implies that $\tilde{H}_*(F;Z) = 0$. Hence F is contractible.

We say that a connected finite CW complex is *prime* if it can never be the total space of a non trivial compact fibration. There has been recent interest in compiling a list of prime CW complexes. Some primes are RP^{2n}, CP^{2n}, QP^n for $n > 1$, S^{2n}, the Cayley plane, and $U(4)/U(2) \times U(2)$. The techniques used are the tranfer for fibrations as in [6] and more recently, the use of Sullivan's minimum models by S. Halperin, "Rational fibrations, minimal models, and Fiberings of homogeneous spaces" to appear in the Transactions of the American Mathematical Society. The following result gives an easy method for constructing Prime CW complexes.

COROLLARY 7. *Let X be a connected finite CW complex. Suppose that $\tilde{H}_*(X;Q) \not\cong 0$. Then if $\chi(\Sigma X) \neq 0$, it follows that ΣX is prime provided one of the following conditions are true:*

(i) *$\chi(\Sigma X)$ is relatively prime to the torsion of $H_*(\Sigma X;Z)$.*

(ii) *$H_*(\Sigma X;Z)$ has no torsion.*

(iii) *The smallest positive integer ℓ such that $H_\ell(\Sigma X;Q) \neq 0$ is ~~odd~~ even.*

PROOF. Let $F \xrightarrow{i} \Sigma X \xrightarrow{p} B$ be a nontrivial compact fibration such that $\tilde{H}_*(\Sigma X;Q) \neq 0$ and $\chi(\Sigma X) \neq 0$ where X is connected. We shall investigate this situation.

Since $0 \neq \chi(\Sigma X) = \chi(F) \cdot \chi(B)$, we see by Corollary 6 that $\tilde{H}_*(B;Z)$ is torsion. Also $\pi_1(F) = \pi_1(\Sigma X) = \pi_1(B) = 0$ as in the proof of Corollary 6. Let B be k-1 connected. Then for some prime p, we have $H_k(B;Z_p) \neq 0$. The Serre spectral sequence for the fibration $F \to \Sigma X \to B$ then reveals that $H_*(\Sigma X;Z)$ must have an element of p-torsion. Now $H_i(\Omega B;Z_p) \neq 0$ for infinitely many i's. Hence Theorem 5 tells us that $d_*: H_*(\Omega B;Z_p) \to H_*(F;Z_p)$ is onto. But by Theorem 12 below, $\chi(F) d_* = 0$. The fact that $\tilde{H}_*(B;Q) = 0$ implies that $\tilde{H}_*(F;Q) \cong \tilde{H}_*(\Sigma X;Q) \neq 0$. Thus $\tilde{H}_*(F;Z_p) \neq 0$. Hence p must divide $\chi(F) = \chi(\Sigma X)$, thus proving (i).

Since $\Omega B \xrightarrow{d} F \xrightarrow{i} \Sigma X$ is a principal fibration, there is an "action" $\hat{d}: F \times \Omega B \to F$. An element $x \in \tilde{H}_*(F;Z_p)$ is said to be *decomposable* if $x = \hat{d}_*(\Sigma x_i \otimes y_i)$ where the dimensions of the y_i are greater than zero. From the argument of Theorem 2 of [10] and the fact that $\chi(F) \neq 0$, we know that the image of d_* does not contain any odd dimensional indecomposable elements. Since d_* is onto, $H_*(F;Z_p)$ does not contain any odd dimensional indecomposable elements.

Now if ℓ is the smallest positive integer such that $H_\ell(F;Z_p) \neq 0$, then any $x \in H_\ell(F;Z_p)$ is indecomposible. Furthermore, if x arises from a torsion element in $H_\ell(F;Z)$, then there is an element $y \in H_{\ell+1}(F;Z_p)$ related to x. It follows from the Bockstein spectral sequence that y is indecomposable. Since either x or y is odd dimensional, it follows that $H_\ell(F;Z)$ is not all torsion and that ℓ is even. Since $H_*(F;Q) \cong H_*(\Sigma X;Q)$, it follows that ℓ is the smallest dimension such that $\tilde{H}_\ell(\Sigma X;Q) \neq 0$, thus proving (iii).

Note that Example C of §2 shows that the condition $\tilde{H}_*(\Sigma X;Q) \neq 0$ is necessary. Whether conditions (i) or (ii) or (iii) are necessary is unknown at this time. As an example of the result above, we propose the following fact: Suppose X is a closed surface, then ΣX is not prime if and only if X is S^2.

COROLLARY 8. *Suppose $F \to \Sigma X \to B$ is a fibration which admits a cross-section. If F is a finite CW complex and $\tilde{H}_i(\Omega B;K) \neq 0$ for all fields K, and some $i > 0$, then the fibration is trivial.*

PROOF. Assume that B is not contractible. We must show that F is contractible. The cross-section $s: B \to \Sigma X$ gives rise to a cross-section $(\Omega s): \Omega B \to \Omega(\Sigma X)$. Hence

$(\Omega p)_*$: $H_*(\Omega\Sigma X;K) \to H_*(\Omega B;K)$ is onto for all fields K, and thus d_*: $H_*(\Omega B;K) \to H_*(F;K)$ is trivial. Also note that the existence of the cross-section implies that F is connected.

Now the pullback fibration $F \to \Sigma Y \to \widetilde{B}$ induced by the covering projection $\widetilde{B} \to B$ satisfies the hypotheses of Theorem 5. Hence d_* is onto. Since $d_* = 0$, we see that $\widetilde{H}_*(F;K) = 0$ for every field K, and since F is compact, $\widetilde{H}_*(F;Z) \cong 0$. Hence F is contractible since $\pi_1(F) = 0$, as may easily be seen from Theorem 4.

COROLLARY 9. *If f: $\Sigma X \to S^n$ has a compact homotopy theoretic fibre, then either f is a homotopy equivalence or $n = 1,2,4,8$ and $\Sigma X = S^1, S^3, S^7, S^{15}$ respectively.*

PROOF. Suppose $F \xrightarrow{i} \Sigma X \xrightarrow{f} S^n$. Then by Theorem 3 $(\Omega f)_*$: $\widetilde{H}_*(\Omega\Sigma X) \to \widetilde{H}_*(\Omega S^n)$ is 1-1 for any field of coefficients. Hence $H_*(\Omega\Sigma X)$ is a subring of $H_*(\Omega S^n)$ = polynomial algebra on λ_{n-1}. Hence $H_*(\Omega\Sigma X)$ must be a polynomial algebra on one generator. Hence $H_*(\Sigma X)$ must have one generator. This implies that ΣX is homotopy equivalent to a sphere. The only non-trivial fibrations of a sphere by a theorem of Browder [5], are $S^1 \to S^3 \to S^2$, $S^3 \to S^7 \to S^4$ and $S^7 \to S^{15} \to S^8$, or a covering map p: $S^1 \to S^1$.

COROLLARY 10. *Let $\Sigma Y \xrightarrow{i} \Sigma X \xrightarrow{p} B$ be a fibration where ΣY is a compact CW complex. Suppose that for every prime p there is an $i > 0$ such that $H_i(\Omega B;Z_p) \neq 0$. Then ΣY is homotopy equivalent to an odd dimensional sphere.*

PROOF. As usual we may assume that B is simply-connected, since if not we could look at the pullback over $\widetilde{B} \to B$. Then by Theorem 5, d_*: $H_*(\Omega B;K) \to H_*(\Sigma Y;K)$ is onto. By Theorem 5 of [10], if $d_* \neq 0$, then ΣY is an odd dimensional rational-homology sphere and a Z_p-homology sphere for all primes p. Since ΣY is compact, this implies that ΣY is a homology sphere. If we consider $\pi_1(\Sigma Y)$, we easily see that $\pi_1(\Sigma Y) = 0$ or $\pi_1(\Sigma Y) = Z$. Then a standard argument results in the fact that ΣY is homotopy equivalent to an odd dimensional sphere if $\pi_1(\Sigma Y) = 0$. If $\pi_1(\Sigma Y) \cong Z$, then Y has two path components Y_1 and Y_2. Since $\widetilde{H}_*(\Sigma Y) \cong Z$, we see that ΣY_1 and ΣY_2 have trivial homology and hence are contractible. Hence ΣY is homotopy equivalent to S^1.

§5. **Suspensions and the evaluation map.** By an evaluation map we mean a map ω: $X \to Y$ such that ω: $X \times * \to X \times Y \xrightarrow{\hat{\omega}} Y$ for some "affiliated" map $\hat{\omega}$ where $\hat{\omega}|* \times Y = 1_Y$. In other words, ω: $X \to (Y^Y,1) \xrightarrow{\omega_*} Y$, that is ω maps through the space of self homotopy equivalences of Y and then is followed by the evaluation at the base point, $\omega_*(f) = f(*)$. Any such map will always be denoted by "ω" in the same manner as "i" almost always denotes an inclusion map.

Examples of evaluation maps are

(i) Fibre inclusion i: $G \to E$ in a principal fibration $G \to E \to B$.

(ii) The connecting map $\Omega B \xrightarrow{d} F$ in the Puppe sequence of a fibration $F \to E \to B$.

(iii) The coset map of a group G and a closed subgroup H, ρ: $G \to G/H$.

(iv) Any map f such that $f = \omega \circ g$ for some evaluation map ω.

The purpose of this section is the study of compact fibrations, $F \to \Sigma X \xrightarrow{\omega} B$, where the projection map is an evaluation map. There is a nontrivial example of this phenomenon, namely the Hopf fibration $S^1 \to S^3 \xrightarrow{\rho} S^2 = S^3/S^1$. We shall show that this is the *only* non-trivial example.

The motivation behind this result is amusing and instructive. We have the following pair of theorem, both the consequence of the transfer for fibrations:

THEOREM 11. [6]. *Let $F \to \Sigma X \xrightarrow{\alpha} B$ be a compact fibration, where $H_*(B;Q) \not\cong 0$. Then $[\alpha]$ has infinite order in the group $[\Sigma X,B]$.*

THEOREM 12. [2]. *Suppose ω: $X \to B$ where X is a finite dimensional and B is a compact CW complex. Then $\chi(B)\{\omega\} = 0$ in the group $\{X,B\}$ of stable maps.*

If we combine Theorem 11 and Theorem 12, we see that $[\omega] \in [\Sigma X,B]$ has infinite order and $\{\omega\} \in \{\Sigma X,B\}$ has finite order if $\chi(B) \neq 0$. Thus one finds, for the Hopf fibration, the well-known fact that $[\rho] \in \pi_3(S^2)$ has infinite order and $2\{\rho\} = 0 \in \pi_4(S^3)$.

It is natural to ask for more non-trivial examples of this phenomenon. What the following theorem shows is that there are none. First we need some lemmas.

LEMMA 13. *Let ω: $X \to Y$ be an evaluation map. Then $(\Omega\omega)_*$: $H_*(\Omega X) \to H_*(\Omega Y)$ maps the Pontrajagin ring of $H_*(X)$ into the center of the Pontrajagin ring of $H_*(Y)$.*

PROOF. By hypothesis there is a map $\hat{\omega}$: $X \times Y \to Y$ such that $\hat{\omega}|X \times * = \omega$ and

$\hat{\omega}|* \times Y =$ identity. Now $(\Omega\hat{\omega})$: $\Omega(X \times Y) = \Omega(X) \times \Omega(Y) \to \Omega Y$ induces a ring homomorphism on the homology groups.

Let $x \in H_*(\Omega X)$ and $y \in H_*(\Omega Y)$. Then $x \times y \in H_*(\Omega X \times \Omega Y)$ may be written in the following forms: $x \times y = (x \times 1)\cdot(1 \times y) = \pm(1 \times y)\cdot(x \times 1)$ where $\cdot$ denotes multiplication in the Pontrajagin ring $H_*(\Omega X \times \Omega Y)$.

Now $(\Omega\hat{\omega})_*(x \times y) = (\Omega\hat{\omega})_*((x \times 1)\cdot(1 \times y)) = ((\Omega\hat{\omega})_*(x \times 1))\cdot((\Omega\hat{\omega})_*(1 \times y)) = ((\Omega\omega)_*(x))\cdot y$ and also $(\Omega\hat{\omega})_*(x \times y) = \pm(\Omega\hat{\omega})_*((1 \times y)\cdot(x \times 1)) = \pm y\cdot((\Omega\omega)_*(x))$. Hence $\pm y\cdot((\Omega\omega)_*(x)) = ((\Omega\omega)_*(x))\, y$ holds for all y. Thus $(\Omega\omega)_*(x)$ is in the center of $H_*(Y)$.

LEMMA 14. *Suppose* $F \xrightarrow{i} \Sigma X \xrightarrow{\omega} B$ *is a compact fibration. Then* $H_i(\Omega B;K)$ *is nonzero for infinitely many i's and some field K if and only if B is not aspherical.* (We do not need a compact F, but F must be connected.)

PROOF. Suppose that B is a $K(\pi,1)$. Then ΩB has the homotopy type of a discrete set of points, hence $H_i(\Omega B;K) = 0$ is zero for all $i > 0$.

Now assume that B is not a $K(\pi,1)$. Then $\pi_i(B) \neq 0$ for some $i > 0$. Since ω_*: $\pi_1(\Sigma X) \to \pi_1(B)$ is onto, B is strongly simple in the sense of Spanier ([16], Example 18 on page 510). Thus by ([16], Theorem 20 on page 510), since B is a finite CW complex and is strongly simple, the homotopy groups $\pi_i(B)$ are finitely generated. Thus the homotopy groups, $\pi_*(\Omega B)$, are finitely generated. Again by Theorem 20 of [16] and the fact that H-spaces are strongly-simple, the integral homology group of the constant component of ΩB are finitely generated. The constant component of ΩB is homeomorphic to $\Omega\tilde{B}$ where $\tilde{B}$ is the universal covering of B. Thus there is, for some i, a finitely generated $H_i(\Omega\tilde{B};Z) \neq 0$. Hence for some field K, $H_i(\Omega\tilde{B};K) \neq 0$. Now applying the Serre spectral sequence to the fibration $\Omega\tilde{B} \to P \to \tilde{B}$ and using the fact that $\tilde{B}$ is finite dimensional and that $\tilde{H}_*(\tilde{B};Z) \neq 0$ and is finitely generated, and hence that $\tilde{H}_*(\tilde{B};K) \neq 0$, we see that $H_i(\Omega\tilde{B};K) \neq 0$ for infinitely many i's. Hence $H_i(\Omega B;K)$ is not zero for infinitely many i's.

LEMMA 15. *If* $F \to \Sigma X \xrightarrow{\omega} B$ *is a compact fibration and if* $H_i(\Omega B;K) \neq 0$ *for infinitely many i's, then*

$$\tilde{H}_*(\Sigma X;K) \cong K.$$

PROOF. First assume X is connected. By Theorem 3, $(\Omega\omega)_*$: $H_*(\Omega\Sigma X;K) \to H_*(\Omega B;K)$ is injective. By Lemma 13 the image of $(\Omega\omega)_*$ lies in the center of the Pontrajagin ring $H_*(\Omega B;K)$ as a subring. Thus $H_*(\Omega\Sigma X;K)$ is an abelian ring. By Theorem 2, the Bott-Samelson theorem, $H_*(\Omega\Sigma X;K)$ is the free tensor algebra on the generators of $\tilde{H}_*(X;K)$, so the requirement that $H_*(\Omega\Sigma X;K)$ is abelian implies that $\tilde{H}_*(X;K) \cong K$. Now if X is not connected, then as in Theorem 4, we have the pullback of the universal covering $F \to \Sigma Y \xrightarrow{\tilde{p}} \tilde{B}$, and ΣY is the universal cover of ΣX. Now $(\Omega\tilde{p})_*$: $H_*(\Omega\Sigma Y;K) \to H_*(\Omega\tilde{B};K)$ is injective by the main theorem. Since $\Omega\tilde{B}$ is homeomorphic to the constant path component of ΩB for any universal covering, we may regard $H_*(\Omega\Sigma Y;K)$ and $H_*(\Omega\tilde{B};K)$ as subrings of $H_*(\Omega\Sigma X;K)$ and $H_*(\Omega B;K)$ respectively and $(\Omega\tilde{p})_*$ as the restriction of $(\Omega\omega)_*$ to $H_*(\Omega\Sigma X;K)$. Hence $H_*(\Omega\Sigma Y;K)$ must be an abelian ring since it is mapped injectively into the center of $H_*(\Omega\tilde{B};K) \subset H_*(\Omega B;K)$. Hence $\tilde{H}_*(Y;K)$ has only one generator at most.

By Theorem 4, ω_*: $\pi_1(\Sigma X) \to \pi_1(B)$ is an isomorphism. However the image of ω_* lies in the center of $\pi_1(B)$, hence $\pi_1(\Sigma X)$ is isomorphic to Z. Thus X consists of two path components. Now Y must be the one point union of infinitely many copies of X_1 and X_2, the path components of X. Since $\tilde{H}_*(\Sigma Y;K) \cong K$ or 0, this implies that $\tilde{H}_*(X_1;K) \cong \tilde{H}_*(X_2;K) \cong 0$, so $H_1(\Sigma X;K) \cong K$ and $H_i(\Sigma X;K) \cong 0$ for $i > 1$.

THEOREM 16. *Consider a fibration* $F \xrightarrow{i} \Sigma X \xrightarrow{\omega} B$ *where* ω *is an evaluation map and F and B are compact connected CW complexes. Then either*

(a) *B is contractible; or*

(b) *F is contractible and* ΣX *is either*

$$S^1 \text{ or } S^3 \text{ or } S^7\text{; or}$$

(c) $F = S^1$, $B = S^2$ *and* $\Sigma X = S^3$.

PROOF. Let K be a field. This proof will have nine cases to track down.

A. Assume that $H_i(\Omega B;K) \neq 0$ for only a finite number of i. Then by Lemma 14 we know that B is an aspherical space. Now $\pi_1(\Sigma X)$ is a finitely generated free group, say of rank r.

A_1. Assume $r = 0$. Then $\pi_1(\Sigma X) = 0$ and hence $\pi_1(B) = 0$ by the homotopy exact sequence. Since B is aspherical, it must be contractible, thus we have conclusion (a).

A_2. Assume $r = 1$. Then $\pi_1(\Sigma X) \cong Z$. Since B is a compact $K(\pi,1)$, $\pi_1(B)$ is torsion free. Thus from the fibre homotopy exact sequence we see that $\pi_1(F) = 0$ and that F is homotopy equivalent to the universal covering space of ΣX. Since $H_1(\Sigma X) \cong Z$, we see that X is the disjoint union of two connected spaces X_1 and X_2. Now if $H_i(\Sigma X_1,Z) \neq 0$ for $i > 0$, then the universal covering, F, of ΣX would have infinitely generated homology and hence could not be a compact CW complex. Thus the integer homology of ΣX_1, and similarly ΣX_2, must be trivial. Hence ΣX_1 and ΣX_2 are both contractible, hence ΣX is homotopy equivalent to S^1. Then it follows that F is contractible and we have arrived at conclusion (b).

A_3. Assume that $r > 1$, so that $\pi_1(\Sigma X)$ is a free group of rank $r > 1$. Since B is a compact aspherical CW complex, $\pi_1(B)$ has no torsion. Also, since $\omega_*: \pi_1(\Sigma X) \to \pi_1(B)$ is onto and since the image of ω_* is in the center of $\pi_1(B)$, we know that $\pi_1(B)$ is abelian. Thus $\pi_1(B)$ is a finitely generated free abelian group. Hence $\pi_1(F)$ cannot be finitely generated and so F cannot be a finite CW complex. It is easy to see that $\pi_1(F)$ is not finitely generated. The exact sequence

$$0 \to \pi_1(F) \to \pi_1(\Sigma X) \to \pi_1(B) \to 0$$

gives rise to a fibration of $K(\pi,1)$'s,

$$K(\pi_1(F),1) \to K(\pi_1(\Sigma X),1) \to K(\pi_1(B),1).$$

Now $K(\pi_1(\Sigma X),1)$ is homotopy equivalent to the one point union of r circles, since $\pi_1(\Sigma X)$ is free of rank r, which has Euler Poincare number equal to 1-r. Also $K(\pi_1(B),1)$ is homotopy equivalent to a torus if $\pi_1(B) \neq 0$ and hence has Euler Poincare number equal to 0. (If $\pi_1(B) = 0$ then B would be contractible since B is aspherical and we would reach conclusion (a).) The product formula for the Euler Poincare numbers for compact fibrations tells us that $0 \neq 1 - r = \chi(F) \cdot 0$ which is impossible. Hence $K(\pi_1(F),1)$ cannot be homotopy equivalent to a compact CW complex. But $\pi_1(F)$ is a subgroup of a free group and hence is a free group, so it must have infinite rank.

B. Assume that $H_i(\Omega B;K) \neq 0$ for infinitely many i. Now by Lemma 15, $\tilde{H}_*(\Sigma X,K) \cong K$.

B_1. Now assume that $\tilde{H}_*(\Sigma X;Q) \cong 0$. Then $\pi_1(\Sigma X) = 0$ and also $\chi(\Sigma X) = 1$ and hence $\chi(F)$ and $\chi(B)$ are ± 1. Since $\chi(Y)\omega_* = 0$ for all evaluation maps onto compact

CW complexes Y, we see that $\omega_*: \tilde{H}_*(\Sigma X;Z) \to \tilde{H}_*(B;Z)$ and $d_*: \tilde{H}_*(\Omega B;Z) \to H_*(F;Z)$ are both trivial homomorphisms. Now let k be the smallest positive integer such that $H_k(B;Z) \neq 0$. Note that $k > 1$ since $\pi_1(B) = 0$ in this case since $\pi_1(\Sigma X) = 0$. Then the Serre exact sequence of the fibration applies and gives us

$$H_k(\Sigma X;Z) \xrightarrow{\omega_*} H_k(B;Z) \xrightarrow{t} H_{k-1}(F;Z).$$

Now $\omega_* = 0$ and $t = 0$ since the transgression t must factor through d_*, [11; Theorem 4] and $d_* = 0$. Hence $H_k(B;Z) = 0$. Thus $\tilde{H}_*(B;Z) = 0$ and so B is contractible since $\pi_1(B) = 0$. This is conclusion (a).

B_2. We assume that $\tilde{H}_*(\Sigma X;Q) \neq 0$. Then ΣX must be a Q-homology sphere and a Z_p-homology sphere for all prime p. Since ΣX is a finite CW complex, $H_*(\Sigma X;Z)$ is finitely generated so ΣX must be a Z-homology sphere. A standard argument shows that ΣX is actually homotopy equivalent to a sphere S^n. Since Corollary 6 implies that $\chi(\Sigma X) = 0$, we see that S^n is actually an odd dimensional sphere.

Now W. Browder [5] proved that every fibration of a sphere with F connected and B not contractible must have fibre either a point, or S^1 or S^3 or S^7.

B_{2a}. Suppose F is contractible. Then the fact that $\omega: S^n \to S^n$ is an evaluation map and a homotopy equivalence implies that S^n is an H-space, and hence by [1] ΣX is either S^1, S^3 or S^7. This is conclusion (b).

B_{2b}. Suppose F is not contractible and ΣX is an odd sphere not of dimension 1, 3 or 7. Then the Whitehead product $[\iota_n,\iota_n] \neq 0$, since otherwise S^n would be an H-space. Now $i: F \to S^n$ is homotopic to a constant map, so from the homotopy exact sequence $\omega_*: \pi_i(S^n) \to \pi_i(B)$ is injective. Thus $0 \neq \omega_*([\iota_n,\iota_n]) = [\omega,\omega]$. But the Whitehead product of any element of $\pi_i(B)$ with an element in the image of an evaluation map must vanish, [9], which is a contradiction.

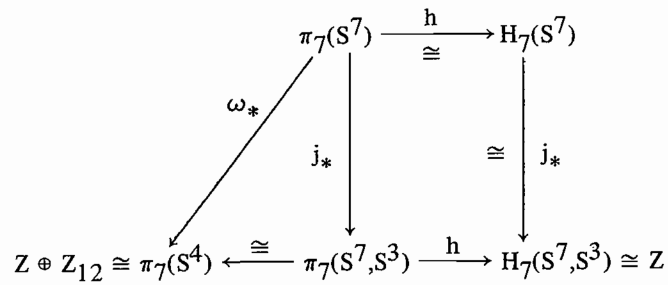
B_{2c}. Suppose that F is not contractible and ΣX has the homotopy type of S^7. Then F is S^1 or S^3 or S^7. Now $F = S^7$ implies that B is contractible by Browder [5].

B_{2ci}. Suppose $F = S^1$. Then by [5] we are considering $S^1 \to S^7 \xrightarrow{\omega} CP^3$. Now from the fibre homotopy exact sequence $[\omega] \in \pi_7(CP^3) \cong Z$ is a generator. It follows from [13] that $\{\omega\} \in \pi_7^S(CP^3) \cong Z_{24}$ is a generator. Hence $\{\omega\}$ has order 24. But $0 = \chi(CP^3)\{\omega\} = 4\{\omega\}$ by Theorem 12.

B_{2cii}. Suppose that $F = S^3$. Then we have a fibration of the form

$$S^3 \to S^7 \xrightarrow{\omega} S^4.$$

Now $\pi_7(S^4) \cong Z \oplus Z_{12}$. Let η generate Z and τ generate Z_{12}. Then $[\omega] = k\eta + \ell\tau$ for some integers k and ℓ. Let h be the Hurewicz homomorphism, and j: $S^7 \to (S^7, S^3)$ the inclusion. We have the following commutative diagram



A diagram chase yields

$$[\omega] = \pm\eta + \ell\tau.$$

Since the suspension homomorphism E: $\pi_7(S^4) \cong Z \oplus Z_{12} \to Z_{24} \cong \pi_8(S^5)$ is onto [19], we see that $E(\eta)$ is a generator of Z_{24} and $E(\tau) \in Z_{12} \subset Z_{24}$. Hence $\{\omega\} = \{\pm\eta + \ell\tau\}$ must have order 24. Since

$$2\{\omega\} = \chi(S^2)\{\omega\} = 0,$$

we have a contradiction.

B_{2d}. If $\Sigma X = S^3$, then by Browder's result we must have a fibration of the form $S^1 \to S^3 \xrightarrow{\omega} S^2$. Thus $[\omega] = \pm\eta$, the Hopf fibration. Since $S^1 \to S^3 \xrightarrow{\rho} S^3/S^1 = S^2$ where ρ is the coset map, the fibration $S^1 \to S^3 \xrightarrow{\omega} S^2$ is the only nontrivial example of $F \to \Sigma X \xrightarrow{\omega} B$.

B_{2e}. Finally if $\Sigma X = S^1$ the only compact fibration with connected fibre F occurs when F is contractible.

REFERENCES

1. Adams, J. F., *On the nonexistence of elements of Hopf invariant one,* Annals. of Math., 72(1960), 20-104.
2. Becker, J. C. and Gottlieb, D. H., *Transfer maps for fibrations and duality,* Composito Mathematica, 33(1976), 107-133.
3. Berstein, I., *On co-groups in the category of graded algebras,* Trans. Amer. Math. Soc., 115(1965), 257-269.

4. Borel, A., *Impossibilité de fibre une sphere par un produit de spheres,* C. R. Acad. Sci., Paris, 231(1950), 943-945.
5. Browder, W., *Higher torsion in H-spaces,* Trans. Amer. Math. Soc., 108(1963), 353-375.
6. Casson, A. and Gottlieb, D. H., *Fibrations with compact fibres,* Amer. J. Math., 99(1977), 159-189.
7. Eckmann, B., Samelson, H., Whitehead, G. W., *On fibering spheres by toruses,* Bull. Amer. Math. Soc., 55(1949), 433-438.
8. Gottlieb, D. H., *A certain subgroup of the fundamental group,* Amer. J. of Math., 87(1965), 840-885.
9. ———, *Evaluation subgroups of homotopy groups,* Amer. J. of Math., 91(1969), 729-756.
10. ———, *The evaluation map and homology,* Michigan Math. J., 19(1972), 289-297.
11. ———, *Witnesses, transgressions, and the evaluation map,* Indiana U. Math. J., 24(1975), 825-836.
12. Halbhavi, I. G. and Varadarajan, K., *Gottlieb sets and duality in homotopy theory,* Can. J. Math., 27(1975), 1042-1055.
13. Mosher, R. E., *Some stable homotopy of complex projective spaces,* Topology, 7(1968), 179-193.
14. Rutter, J. W., *The suspension of the loops on a space with comultiplication,* Math. Ann., 209(1974), 69-82.
15. ———, *The loops on the suspension of a space with multiplication,* Math. Ann., 214(1975), 81-94.
16. Spanier, E., *Algebraic Topology,* McGraw Hill, 1966, New York.
17. Stasheff, J., *H-spaces from a homotopy point of view,* Lecture Notes in Math., 161, Springer Verlag, New York.
18. Sugawara, M., *On a condition that a space is an H-space,* Math. J. Okayama Univ., 6(1957), 109-129.
19. Toda, H., *Composition methods in homotopy groups of spheres,* Annals. of Mathematics Studies, 49, Princeton University Press, Princeton, New Jersey.

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